The Dirichlet problem for p-harmonic functions with respect to the Mazurkiewicz boundary, and new capacities

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Abstract In this paper we develop the Perron method for solving the Dirichlet problem for the analog of the p-Laplacian, i.e. for p-harmonic functions, with Mazurkiewicz boundary values. The setting considered here is that of metric spaces, where the boundary of the domain in question is replaced with the Mazurkiewicz boundary. Resolutivity for Sobolev and continuous functions, as well as invariance results for perturbations on small sets, are obtained. We use these results to improve the known resolutivity and invariance results for functions on the standard (metric) boundary. We also illustrate the results of this paper by discussing several examples.

Key words and phrases: capacity, Dirichlet problem, doubling measure, finite connectivity at the boundary, inner metric, Mazurkiewicz distance, metric space, nonlinear potential theory, perturbation invariance, Poincaré inequality, p-energy minimizer, p-harmonic function, Perron method, resolutive.

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1. Introduction

When considering the Dirichlet problem on the slit disc $B((0,0),1) \setminus [0,1] \subset \mathbf{R}^2$ it is quite natural to allow for two boundary conditions at each point in the slit $(0,1]$ (apart from the tip), one from above and one from below.

Our aim in this paper is to give a general approach (via the Mazurkiewicz boundary) suitable for solving this generalized p-harmonic Dirichlet problem for a large class of domains in $\mathbf{R}^n$, as well as in metric spaces. We develop the Perron method for the Mazurkiewicz boundary and obtain resolutivity and invariance results for it, which also improve a number of older results for the given boundary. These results are new even in the (unweighted) Euclidean setting.

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A continuous function u is *p-harmonic*, $1 < p < \infty$, if it locally minimizes the p-energy integral

$$\int |\nabla u|^p \, dx.$$

When $p = 2$ this reduces to the classical harmonic functions, while for $p \neq 2$ it is the main prototype for a nonlinear elliptic equation. To generalize this to metric spaces, the concept of upper gradients (introduced by Heinonen and Koskela in [26]) is used, leading to a variational definition similar to the one above. In such a general setting, there is no corresponding equation. The nonlinear potential theory associated with p-harmonic functions has been studied for half a century, first on $\mathbf{R}^n$ and then in various other situations (manifolds, Heisenberg groups, graphs etc.). The metric space theory is more recent and was first considered by Shanmugalingam [45]. It gives a unified treatment covering most of the earlier cases. For further development see the monographs Heinonen–Kilpeläinen–Martio [25] (for weighted $\mathbf{R}^n$) and Björn–Björn [12] (for metric spaces) and the references therein.

To describe our approach, let Ω be a bounded domain in $\mathbf{R}^n$. We define the *Mazurkiewicz distance* d_M on Ω by

$$d_M(x, y) = \inf_E \operatorname{diam} E,$$

where the infimum is taken over all connected sets $E \subset \Omega$ containing $x, y \in \Omega$. (The Mazurkiewicz distance was first used by Mazurkiewicz [39] in 1916, but goes under different names in the literature, see Remark 4.2.) The *Mazurkiewicz boundary* is then defined using the completion of (Ω, d_M). In the slit disc this leads to the desired boundary with two boundary points corresponding to each point in the slit, while for smooth domains it coincides with the usual boundary. In this paper we study Perron solutions for p-harmonic functions with respect to the Mazurkiewicz boundary.

In the Perron method, given a function f on the boundary, an upper and a lower Perron solution are constructed using superharmonic (or p-superharmonic) functions. When these two solutions coincide they give a reasonable solution to the Dirichlet problem with f as boundary values (for harmonic or p-harmonic functions). The function f is then called *resolutive*, and the solution is denoted by Pf.

The Perron method was introduced by Perron [43] (and independently by Remak [44]) in 1923 for harmonic functions. Wiener and Brelot made important contributions in the linear case (on $\mathbf{R}^n$) leading to Brelot's resolutivity result [17], which says that the resolutive functions are exactly the L^1 functions with respect to harmonic measure. For this reason the method is often called the Perron–Wiener–Brelot method, but since this is less appropriate in the nonlinear situation we prefer to name it just after Perron.

For p-harmonic functions on $\mathbf{R}^n$ the Perron method was first studied by Granlund–Lindqvist–Martio [22]. Kilpeläinen [30] showed that continuous functions are resolutive, and Heinonen–Kilpeläinen–Martio [25] adapted the proof to weighted $\mathbf{R}^n$. Using a different approach this result was further generalized to metric spaces in Björn–Björn–Shanmugaglingam [14]. Therein it was also shown that restrictions of quasicontinuous representatives of Sobolev functions on the entire metric space (e.g. $\mathbf{R}^n$) are resolutive. Moreover, if f is either such a Sobolev function or $f \in C(\partial\Omega)$, and $h = f$ q.e. (i.e. outside a set of capacity zero), then h is also resolutive and $Ph = Pf$. In this paper we improve upon both these results.

To be able to generalize this theory to the Mazurkiewicz boundary, we need the Mazurkiewicz boundary (or rather the Mazurkiewicz closure) to be compact (although, the paper Estep–Shanmugalingam [19] avoids the compactness requirement for the prime end boundary). It was shown by Karmazin [29] that this happens

if and only if the domain is finitely connected at the boundary (for a more elementary and self-contained proof of this fact see Björn–Björn–Shanmugaglingam [16]). Under this assumption we generalize all the results from [14] mentioned above to the Mazurkiewicz boundary, with some additional improvements. To do so we consider a new relative capacity, $\overline{C}_p$, adapted to the topology that connects the domain to its Mazurkiewicz boundary. With this new capacity, we consider how the boundary looks from inside Ω, not from the underlying metric space or even within the closure of Ω.

Using the new capacity we also improve upon the results in [14] when considering the usual Perron solutions with respect to the given metric. In the invariance results there, we replace the usual Sobolev capacity with the smaller capacity $\overline{C}_p$, thus allowing for perturbations on larger sets. We also obtain resolutivity for more functions.

For harmonic functions on the slit disc there are two classical approaches: the prime end boundary introduced by Carathéodory [18] in 1913 and the Martin boundary introduced by Martin [38] in 1941.

The minimal Martin kernel, a generalization of the Poisson kernel, is built for the harmonic equation, and gives a very well suited boundary for the linear harmonic Dirichlet problem. First proposed by Martin [38], this notion was developed further by many, including Ancona [3], [4] and Anderson–Schoen [5]. There are p-harmonic generalizations of the Martin boundary, see e.g. Lewis–Nyström [37], but unlike in the linear case $p = 2$, these generalizations are not connected to integral representations of solutions to the Dirichlet problem.

The prime end boundary is instead constructed directly from the geometry of the domain and does not rely on any underlying equation. Carathéodory's original approach works very well for simply (and finitely) connected planar domains. Over the years there have been many suggestions for extending prime ends to more general situations, by different people and with different applications in mind, see the discussion in [1]. Recently Adamowicz–Björn–Björn–Shanmugalingam [1] gave a definition of prime end boundary suitable for a large class of domains in metric spaces. For domains which are finitely connected at the boundary the Mazurkiewicz boundary is homeomorphic to the prime end boundary of [1], see [1, Corollary 10.9]. Thus our results with respect to the Mazurkiewicz boundary can equivalently be formulated for the prime end boundary in such domains. In the special case of the topologist's comb, further improvements upon the general results of this paper are given in A. Björn [9].

The outline of the paper is as follows: After giving a survey of background results from first-order analysis on metric spaces in Section 2, we introduce the new capacity $\overline{C}_p$ in Section 3 and the Mazurkiewicz distance in Section 4. Section 5 is devoted to Sobolev spaces with respect to the Mazurkiewicz distance. The necessary background theory on p-harmonic and superharmonic functions is given in Section 6, making it possible to define Perron solutions with respect to the Mazurkiewicz boundary in the subsequent section.

Our main resolutivity result is given in Theorem 7.4. In Section 8 we use Theorem 7.4 to show resolutivity of continuous functions and to obtain invariance results for perturbations along the lines indicated above. New resolutivity and invariance results with respect to the given metric are described in Section 9, and some further generalizations of the results in Sections 7 and 8 are given in Section 11. Section 10 is devoted to a number of examples showing how our results can be applied. We end the paper with an appendix comparing the different capacities used in this paper.

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2. Notation and preliminaries

We assume throughout the paper that $1 \le p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$ (we adopt the convention that balls are nonempty and open). We emphasize that the σ-algebra on which μ is defined is obtained by the completion of the Borel σ-algebra. It follows that X is separable.

We also assume that $\Omega \subset X$ is a nonempty open set. Further standing assumptions will be given at the end of Section 3 and at the beginning of subsequent sections.

A *curve* is a continuous mapping from an interval, and a *rectifiable* curve is a curve with finite length. We will only consider curves which are nonconstant, compact and rectifiable. A curve can thus be parameterized by its arc length ds.

We follow Heinonen and Koskela [26] in introducing upper gradients as follows (they called them very weak gradients).

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all (nonconstant, compact and rectifiable) curves $\gamma : [0, l_\gamma] \to X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\, ds, \tag{2.1}$$

where we follow the convention that the left-hand side is ∞ whenever both terms therein are infinite. If g is a nonnegative measurable function on X and if (2.1) holds for p-almost every curve (see below), then g is a *p-weak upper gradient* of f.

Here we say that a property holds for *p-almost every curve* if it fails only for a curve family Γ with zero p-modulus, i.e. there exists $0 \le \rho \in L^p(X)$ such that $\int_\gamma \rho\, ds = \infty$ for every curve $\gamma \in \Gamma$. Note that a p-weak upper gradient *need not* be a Borel function, only measurable. Given that the underlying measure μ is Borel regular, every measurable function g can be modified on a set of measure zero to obtain a Borel function, from which it follows that $\int_\gamma g\, ds$ is defined (with a value in $[0, \infty]$) for p-almost every curve γ. For proofs of these and all other facts in this section we refer to Björn–Björn [12] and Heinonen–Koskela–Shanmugalingam–Tyson [27]. (Some of the references we mention below may not provide a proof in the generality considered here, but such proofs are given in [12].)

The p-weak upper gradients were introduced in Koskela–MacManus [35]. It was also shown there that if $g \in L^p_{\mathrm{loc}}(X)$ is a p-weak upper gradient of f, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j - g \to 0$ in $L^p(X)$. If f has an upper gradient in $L^p_{\mathrm{loc}}(X)$, then it has a *minimal p-weak upper gradient* $g_f \in L^p_{\mathrm{loc}}(X)$ in the sense that for every p-weak upper gradient $g \in L^p_{\mathrm{loc}}(X)$ of f we have $g_f \le g$ a.e., see Shanmugalingam [46] and Hajłasz [23]. The minimal p-weak upper gradient is well defined up to a set of measure zero in the cone of nonnegative functions in $L^p_{\mathrm{loc}}(X)$. Following Shanmugalingam [45], we define a version of Sobolev spaces on the metric space X.

Definition 2.2. Whenever $f \in L^p(X)$, let

$$\|f\|_{N^{1,p}(X)} = \left(\int_X |f|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of f. The *Newtonian space* on X is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [45]. In this paper we assume that functions are defined everywhere, not just up to an equivalence class in the corresponding function space. When we say that $f \in N^{1,p}(X)$ we thus assume that f is a function defined everywhere. We say that $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if for every $x \in \Omega$ there exists r_x such that $B(x, r_x) \subset \Omega$ and $f \in N^{1,p}(B(x, r_x))$.

If $f, h \in N^{1,p}_{\mathrm{loc}}(X)$, then $g_f = g_h$ a.e. in $\{x \in X : f(x) = h(x)\}$, in particular $g_{\min\{f,c\}} = g_f \chi_{f<c}$ for $c \in \mathbf{R}$. For these and other facts on p-weak upper gradients, see, e.g., Björn–Björn [10], Section 3.

Definition 2.3. Let $\Omega \subset X$ be an open set. The (Sobolev) *capacity* (with respect to Ω) of a set $E \subset \Omega$ is the number

$$C_p(E; \Omega) = \inf_u \|u\|^p_{N^{1,p}(\Omega)},$$

where the infimum is taken over all $u \in N^{1,p}(\Omega)$ such that $u = 1$ on E.

Observe that the above capacity is *not* the so-called variational capacity, which is obtained by minimizing $\|g_u\|^p_{L^p(\Omega)}$ over $u \in N^{1,p}_0(\Omega)$ such that $u = 1$ on E.

For a given set E we will consider the capacity taken with respect to different sets Ω. When the capacity is taken with respect to the underlying metric space X, we usually drop X from the notation and merely write $C_p(E)$. The capacity is countably subadditive. For this and other properties as well as equivalent definitions of the capacity we refer to Björn–Björn [12].

We say that a property holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has capacity zero. When needed, we shall specify the capacity with respect to which q.e. is taken. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [45] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

We now introduce the space of Newtonian functions with zero boundary values as follows:

$$N^{1,p}_0(\Omega; A) = \{f|_\Omega : f \in N^{1,p}(A) \text{ and } f = 0 \text{ in } A \setminus \Omega\},$$

where $A \subset X$ is a measurable set containing Ω. (In Section 5, this definition will be applied also to Ω equipped with the Mazurkiewicz distance d_M, and then A will be replaced by the Mazurkiewicz closure $\overline{\Omega}^M$ of Ω with respect to this metric.) As with the capacity, when $A = X$ we usually drop X from the notation and merely write $N^{1,p}_0(\Omega)$. It is fairly easy to see that $N^{1,p}_0(\Omega) = N^{1,p}_0(\Omega; \overline{\Omega})$, see Björn–Björn [12]. One can also replace the assumption "$f = 0$ on $A \setminus \Omega$" with "$f = 0$ q.e. on $A \setminus \Omega$" without changing the obtained space $N^{1,p}_0(\Omega; A)$. Functions from $N^{1,p}_0(\Omega; A)$ can be defined to be zero q.e. in $A \setminus \Omega$ and we will regard them in that sense if needed. Here q.e. is taken with respect to the ambient set A.

We say that μ is *doubling* if there exists a *doubling constant* $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X,

$$0 < \mu(2B) \le C\mu(B) < \infty,$$

where $\lambda B = B(x_0, \lambda r)$.

Definition 2.4. We say that X supports a *p-Poincaré inequality* if there exist constants $C>0$ and $\lambda \ge 1$ such that for all balls $B \subset X$, all integrable functions f on X and all upper gradients g of f,

$$\fint_B |f-f_B|\,d\mu \le C(\operatorname{diam} B)\biggl(\fint_{\lambda B} g^p\,d\mu\biggr)^{1/p}, \tag{2.2}$$

where $f_B := \fint_B f\,d\mu := \int_B f\,d\mu/\mu(B)$.

In the definition of Poincaré inequality we can equivalently assume that g is a p-weak upper gradient—see the comments above.

If X is complete and supports a p-Poincaré inequality and μ is doubling, then Lipschitz functions are dense in $N^{1,p}(X)$, see Shanmugalingam [45], and the functions in $N^{1,p}(X)$ and those in $N^{1,p}(\Omega)$ are *quasicontinuous* (see Theorem 2.5 below), i.e. for every $\varepsilon>0$ there is an open set U such that $C_p(U)<\varepsilon$ and $f|_{X\setminus U}$ is real-valued continuous. This means that in the Euclidean setting, $N^{1,p}(\mathbf{R}^n)$ is the refined Sobolev space as defined in Heinonen–Kilpeläinen–Martio [25, p. 96]; we refer interested readers to [45] for this fact, or to Björn–Björn [12] for a proof of this fact valid in weighted $\mathbf{R}^n$. This is the main reason why, unlike in the classical Euclidean setting, we do not require the functions u in the definition of capacity to be 1 in a neighbourhood of E. Moreover, X is *quasiconvex*, i.e. there is a constant L such that any two points $x,y\in X$ can be connected by a curve of length at most $Ld(x,y)$. This fact was first observed by Semmes. For a proof see Hajłasz–Koskela [24, Proposition 4.4]. Recall also that X is *proper* if all closed bounded subsets of X are compact. If μ is doubling then X is complete if and only if X is proper.

We will need the following results from Björn–Björn–Shanmugalingam [15].

Theorem 2.5. ([15, Theorem 1.1 and Corollary 1.3] or [12, Theorems 5.29 and 5.31]) *Assume that X is proper and that continuous functions are dense in $N^{1,p}(X)$. (This happens if, for example, X is complete with μ doubling and supporting a p-Poincaré inequality.) Then*

(a) *C_p is an* outer capacity*, i.e. for all $E\subset\Omega$,*

$$C_p(E) = \inf_{\substack{G\supset E\\ G \text{ open}}} C_p(G);$$

(b) *every $u\in N^{1,p}_{\mathrm{loc}}(\Omega)$ is quasicontinuous in Ω.*

We do not know if there is any metric measure space X for which there is a nonquasicontinuous function in $N^{1,p}(X)$, nor if there is any X such that C_p is not an outer capacity. However, even if we do not know that continuous functions form a dense subclass of $N^{1,p}(X)$, in proper spaces we have the outer capacity property at the level of capacitary null sets, as the following proposition shows.

Proposition 2.6. ([15, Proposition 1.4] (or [12, Proposition 5.27])) *Let X be proper and let $E\subset X$ with $C_p(E)=0$. Then for every $\varepsilon>0$, there is an open set $U\supset E$ with $C_p(U)<\varepsilon$.*

3. The capacity $\overline{C}_p(\,\cdot\,;\Omega)$

In this section we introduce a new capacity $\overline{C}_p(\,\cdot\,;\Omega)$, which is useful in the study of Perron solutions later in the paper.

Definition 3.1. For $E \subset \overline{\Omega}$ let

$$\overline{C}_p(E;\Omega) = \inf_{u\in\mathcal{A}_E} \|u\|^p_{N^{1,p}(\Omega)},$$

where $u \in \mathcal{A}_E$ if $u \in N^{1,p}(\Omega)$ satisfies both $u \geq 1$ on $E \cap \Omega$ and

$$\liminf_{\Omega\ni y\to x} u(y) \geq 1 \quad \text{for all } x \in E \cap \partial\Omega.$$

For $E \subset \Omega$ the new capacity $\overline{C}_p(E;\Omega)$ equals $C_p(E;\Omega)$. The novelty here is that we extend the "Ω-capacity" to sets in the closure $\overline{\Omega}$. On the closure one may of course consider the capacity $C_p(\,\cdot\,;\overline{\Omega})$, but the new capacity, being smaller (see the appendix), makes some of our results more general. See the appendix for a comparison of various capacities and Section 10 for examples where we obtain better results using the new capacity.

Let us deduce some of the properties of the new capacity $\overline{C}_p$. The properties of subadditivity and outer capacity will be important. By truncation it is easy to see that one may as well take the infimum over all $u \in \widetilde{\mathcal{A}}_E := \{u \in \mathcal{A}_E : 0 \leq u \leq 1\}$.

Proposition 3.2. *Let $E, E_1, E_2, \dots$ be arbitrary subsets of $\overline{\Omega}$. Then*

(i) $\overline{C}_p(\varnothing;\Omega) = 0$;
(ii) $\mu(E \cap \Omega) \leq \overline{C}_p(E;\Omega)$;
(iii) *if $E_1 \subset E_2$, then $\overline{C}_p(E_1;\Omega) \leq \overline{C}_p(E_2;\Omega)$;*
(iv) *$\overline{C}_p(\,\cdot\,;\Omega)$ is countably subadditive, i.e.*

$$\overline{C}_p\biggl(\bigcup_{i=1}^\infty E_i;\Omega\biggr) \leq \sum_{i=1}^\infty \overline{C}_p(E_i;\Omega).$$

Proof. The claims (i)–(iii) are immediate from the definition. We now prove (iv).

Let $\varepsilon > 0$, and choose u_i with $u_i \in \widetilde{\mathcal{A}}_{E_i}$ and upper gradients g_i in Ω such that

$$\|u_i\|^p_{L^p(\Omega)} + \|g_i\|^p_{L^p(\Omega)} \leq \overline{C}_p(E_i;\Omega) + \frac{\varepsilon}{2^i}.$$

Let $u = \sup_i u_i$ and $g = \sup_i g_i$. It is an easy exercise to see that g is an upper gradient of u, see Lemma 1.28 in Björn–Björn [12] for a proof. Clearly $u \in \widetilde{\mathcal{A}}_E$, where $E = \bigcup_{i=1}^\infty E_i$. Hence

$$\overline{C}_p(E;\Omega) \leq \|u\|^p_{N^{1,p}(\Omega)} \leq \int_\Omega \sum_{i=1}^\infty u_i^p \, d\mu + \int_\Omega \sum_{i=1}^\infty g_i^p \, d\mu \leq \sum_{i=1}^\infty \Bigl(\overline{C}_p(E_i;\Omega) + \frac{\varepsilon}{2^i}\Bigr).$$

Letting $\varepsilon \to 0$ completes the proof of (iv). ☐

Proposition 3.3. *Assume that all functions in $N^{1,p}(\Omega)$ are quasicontinuous. Then $\overline{C}_p(\,\cdot\,;\Omega)$ is an outer capacity, i.e. for all $E \subset \overline{\Omega}$,*

$$\overline{C}_p(E;\Omega) = \inf_{\substack{G\supset E \\ G \text{ relatively open in } \overline{\Omega}}} \overline{C}_p(G;\Omega).$$

By Theorem 2.5, we know that Proposition 3.3 applies when X is complete and the measure on X is doubling and supports a p-Poincaré inequality. In Section 8 we will apply this result to the $\overline{C}_p(\,\cdot\,;\Omega^M)$ capacity defined below; this is possible since $N^{1,p}(\Omega^M) = N^{1,p}(\Omega)$.

Proof. The fact that the left-hand side is not larger than the right-hand side follows directly from the monotonicity in Proposition 3.2 (iii).

To prove the converse inequality, let $E \subset \overline{\Omega}$, $0 < \varepsilon < 1$, and $u \in \widetilde{\mathcal{A}}_E$ be such that

$$\|u\|_{N^{1,p}(\Omega)} \le \overline{C}_p(E;\Omega)^{1/p} + \varepsilon.$$

By assumption, u is quasicontinuous in Ω. Hence there is an open set $V \subset \Omega$ with $C_p(V;\Omega)^{1/p} < \varepsilon$ such that $u|_{\Omega\setminus V}$ is continuous. Thus, there is an open set $U \subset \Omega$ such that

$$U \setminus V = \{x \in \Omega : u(x) > 1-\varepsilon\} \setminus V \supset (E \cap \Omega) \setminus V.$$

We can also find $v \ge \chi_V$ with $\|v\|_{N^{1,p}(\Omega)} < \varepsilon$. Let

$$w = \frac{u}{1-\varepsilon} + v.$$

Then $w \ge 1$ on $(U \setminus V) \cup V = U \cup V$, an open set containing $E \cap \Omega$. Moreover, for each $x \in E \cap \partial\Omega$ there is $r_x > 0$ such that

$$u > 1-\varepsilon \quad \text{in } B(x,r_x) \cap \Omega,$$

and hence $w \ge 1$ in $B(x,r_x) \cap \Omega$. Therefore

$$W = U \cup V \cup \bigcup_{x \in E \cap \partial\Omega} (B(x,r_x) \cap \overline{\Omega}).$$

is a relatively open subset of $\overline{\Omega}$ containing E and $w \in \mathcal{A}_W$. Hence

$$\begin{aligned}\overline{C}_p(E;\Omega)^{1/p} &\le \inf_{\substack{G \supset E \\ G \text{ relatively open in } \overline{\Omega}}} \overline{C}_p(G;\Omega)^{1/p} \le \overline{C}_p(W;\Omega)^{1/p} \le \|w\|_{N^{1,p}(\Omega)} \\ &\le \frac{1}{1-\varepsilon}\|u\|_{N^{1,p}(\Omega)} + \|v\|_{N^{1,p}(\Omega)} \le \frac{1}{1-\varepsilon}(\overline{C}_p(E;\Omega)^{1/p} + \varepsilon) + \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ completes the proof. □

For the sake of clarity we make the following explicit definition. We set $\overline{\mathbf{R}} := [-\infty,\infty]$.

Definition 3.4. A function $f \in \overline{\Omega} \to \overline{\mathbf{R}}$ is $\overline{C}_p(\,\cdot\,;\Omega)$-*quasicontinuous* if for every $\varepsilon > 0$ there is a relatively open set $U \subset \overline{\Omega}$ such that $\overline{C}_p(U;\Omega) < \varepsilon$ and $f|_{\overline{\Omega}\setminus U}$ is real-valued continuous.

We assume from now on that X is a complete metric space supporting a p-Poincaré inequality, that μ is doubling, and that $1 < p < \infty$. It follows that X is quasiconvex, and in particular connected and locally connected.

4. The Mazurkiewicz distance d_M

In addition to the standing assumptions mentioned at the end of the previous section, we assume in this section that Ω is a bounded domain, *i.e. a bounded nonempty open connected set.*

Definition 4.1. We define the *Mazurkiewicz distance* d_M on Ω by

$$d_M(x,y) = \inf_E \operatorname{diam} E,$$

where the infimum is over all connected sets $E \subset \Omega$ containing $x, y \in \Omega$. We further define the *inner metric* d_{inner} on Ω by

$$d_{\text{inner}}(x,y) = \inf_\gamma l_\gamma,$$

where the infimum is taken over all curves $\gamma : [0, l_\gamma] \to \Omega$ parameterized by arc length and such that $\gamma(0) = x$ and $\gamma(l_\gamma) = y$.

A consequence of the quasiconvexity of X is that each pair of points $x, y \in \Omega$ can be connected by a rectifiable curve in Ω (see Björn–Björn [12, Lemma 4.38]), and so $d_{\mathrm{inner}}(x,y) < \infty$. Hence both d_{inner} and d_M are metrics on Ω.

Remark 4.2. The Mazurkiewicz distance was introduced by Mazurkiewicz [39], in relation to a classification of points on n-dimensional Euclidean continua. It goes under different names in the literature, and is e.g. denoted ρ_A in [39], called relative distance and denoted ϱ_r in Kuratowski [36], called Mazurkiewicz intrinsic metric and denoted δ_D in Karmazin [29], and called inner diameter distance in Aikawa–Hirata [2], Freeman–Herron [21] and Herron–Sullivan [28]. Here and in Adamowicz–Björn–Björn–Shanmugalingam [1] and Björn–Björn–Shanmugalingam [16] we call it the Mazurkiewicz distance.

Lemma 4.3. *We always have $d \le d_M \le d_{\mathrm{inner}}$. Furthermore, if Ω is L-quasiconvex, then we also have $d_{\mathrm{inner}} \le Ld$.*

Note that even though X is quasiconvex, we will consider these distances with respect to Ω, which, in general, is not quasiconvex.

Proof. The first inequality is obvious. As for the second inequality, let $\gamma : [0, l_\gamma] \to \Omega$ be a curve, parameterized by arc length, such that $\gamma(0) = x$ and $\gamma(l_\gamma) = y$. Then the image $\widehat{\gamma} := \gamma([0, l_\gamma])$ is connected and $\operatorname{diam} \widehat{\gamma} \le l_\gamma$. Hence

$$d(x,y) \le d_M(x,y) = \inf_E \operatorname{diam} E \le \inf_\gamma \operatorname{diam} \widehat{\gamma} \le \inf_\gamma l_\gamma = d_{\mathrm{inner}}(x,y).$$

If Ω is L-quasiconvex, then $l_\gamma \le Ld(x,y)$ for some curve γ in the infimum, proving the third inequality. $\square$

Lemma 4.4. *For a curve $\gamma : [0, l_\gamma] \to \Omega$, arc lengths with respect to d, d_M and d_{inner} are the same.*

Proof. That arc lengths are the same with respect to d and d_{inner} is folklore, for a proof see Björn–Björn [12, Lemma 4.43]. It then follows from Lemma 4.3 that arc length is also the same with respect to d_M. $\square$

In the rest of the paper it will be important for us to work with both the given metric d and the Mazurkiewicz distance d_M. The inner metric d_{inner} will however not be used in the rest of the paper, apart from in some examples in Section 10.

In this paper, by $\overline{\Omega}^M$ we mean the completion of the metric space $\Omega^M := (\Omega, d_M)$, where the Mazurkiewicz distance d_M comes from Ω. On $\overline{\Omega}^M$, d_M always refers to the metric in $\overline{\Omega}^M$ inherited from the metric d_M on Ω. The focus of this paper is to use the Perron method to study solutions of Dirichlet problems with various boundary data with respect to the Mazurkiewicz boundary. For this method to work it will be vitally important that $\overline{\Omega}^M$ is compact. It turns out that the compactness of $\overline{\Omega}^M$ has a very geometric characterization. We state it next, before defining the concepts involved.

Theorem 4.5. *The closure $\overline{\Omega}^M$ is compact if and only if Ω is finitely connected at the boundary.*

This theorem holds whenever X is proper and locally connected. For a proof see Karmazin [29, Theorem 1.3.8] (in Russian) or Björn–Björn–Shanmugalingam [16].

Definition 4.6. We say that Ω is *finitely connected* at $x_0 \in \partial\Omega$ if for every $r > 0$ there is an open (in X) set G such that $x_0 \in G \subset B(x_0, r)$ and $G \cap \Omega$ has only finitely many components.

If there is $N > 0$ such that for every $r > 0$ there is an open (in X) set G such that $x_0 \in G \subset B(x_0, r)$ and such that $G \cap \Omega$ has at most N components, then we say that Ω is *boundedly connected* at x_0. If moreover N is minimal, we say that Ω is *N-connected* at x_0. We say that Ω is *locally connected* at $x_0 \in \partial\Omega$ if it is 1-connected at x_0.

We say that Ω has one of the above properties *at the boundary* if it has that property at each boundary point.

The terminology above follows Näkki [40]. (Näkki [41] has informed us that he learned about the terminology from Väisälä, who however first seems to have used it in print in [48].) For planar domains, the concept of finite connectedness at the boundary was used by Newman [42] (only in the first edition of his book). Beware that the notion of finitely connected domains is a completely different notion; a domain is finitely connected if its fundamental group is finitely generated. Observe also that the balls in the definition above are taken with respect to the given metric d.

If Ω is finitely connected at the boundary, then the map $\Phi : \overline{\Omega}^M \to \overline{\Omega}$ defined below sheds more light on the relation between the Mazurkiewicz boundary $\partial_M\Omega$ and the metric boundary $\partial\Omega$. On Ω^M the map Φ is the natural map given by $\Phi(x) = x$ when $x \in \Omega^M$. This map is a 1-Lipschitz map on Ω^M (since $d \le d_M$), and hence has a unique continuous extension to $\overline{\Omega}^M$, which we again denote by Φ; this extension is also 1-Lipschitz. If $x_0 \in \partial\Omega$ and Ω is N-connected at x_0, then $\Phi^{-1}(x_0)$ consists of exactly N points, while if Ω is finitely but not boundedly connected at $x_0 \in \partial\Omega$, then $\Phi^{-1}(x_0)$ is an infinite countable set. For a more detailed description, see Björn–Björn–Shanmugalingam [16].

5. The Sobolev spaces $N^{1,p}(\Omega^M)$, $N^{1,p}(\overline{\Omega}^M)$

In addition to the standing assumptions described at the end of Section 3, we assume in this section that Ω is a bounded domain which is finitely connected at the boundary.

There are now two different metrics on Ω of interest here: the given metric d and the Mazurkiewicz distance d_M. To make the distinction clear we denote the metric space (Ω, d) by Ω, and the metric space (Ω, d_M) by Ω^M. We equip both of them with the measure μ (or strictly speaking, the restriction $\mu|_\Omega$). As sets, $\Omega = \Omega^M$, and it is only the metrics that are different. It is when taking closures that we really have to distinguish between Ω and Ω^M. Note that $\Omega \subset X$ and $\Omega^M \subset \overline{\Omega}^M$. The closure $\overline{\Omega}$ of Ω in X is compact because X is proper. The closure of Ω^M is the completion $\overline{\Omega}^M$ as introduced in Section 4, which is compact by Theorem 4.5.

We also equip $\overline{\Omega}$ and $\overline{\Omega}^M$ with measures as follows. For $\overline{\Omega}$ the natural choice is to equip it with $\mu|_{\overline{\Omega}}$. However, it is sometimes preferable to equip $\overline{\Omega}$ with $\mu_0 := \mu|_\Omega$ so that $\mu_0(\partial\Omega) = 0$ (strictly speaking we let $\mu_0(E) = \mu(E \cap \Omega)$ for $E \subset \overline{\Omega}$). For $\overline{\Omega}^M$ we have no natural measure on $\partial\Omega$ and we always equip it with (the zero extension of) $\mu|_\Omega$ so that $\mu(\partial_M\Omega) = 0$. Since Ω is an open subset of $\overline{\Omega}^M$, this zero extension is a Borel measure.

The two metrics d and d_M are locally equivalent in Ω, and give the same (not only equivalent) arc lengths of curves. This is important from the point of view of upper gradients and Newtonian spaces, see Lemma 4.4.

Let $f : \Omega \to \overline{\mathbf{R}}$ be an arbitrary (extended real-valued) function on Ω, and let $g : \Omega \to [0, \infty]$. As arc lengths for curves in Ω are the same with respect to d and d_M, g will be an upper gradient of f with respect to Ω if and only if it is an upper gradient with respect to Ω^M. Since we equip both metric spaces with the

same measure we see that $N^{1,p}(\Omega) = N^{1,p}(\Omega^M)$ and $N^{1,p}_{\mathrm{loc}}(\Omega) = N^{1,p}_{\mathrm{loc}}(\Omega^M)$ (with the same (semi)norms). It also follows that g is a p-weak upper gradient of f with respect to Ω if and only if it is a p-weak upper gradient with respect to Ω^M. See also Proposition 5.3 where similar results are obtained for Newtonian functions with zero boundary values.

The following two lemmas relate Newtonian spaces and capacities with respect to Ω, $\overline{\Omega}$ and $\overline{\Omega}^M$. Recall that Φ is the 1-Lipschitz extension to $\overline{\Omega}^M$ of the identity map on Ω^M, see the end of Section 4.

Lemma 5.1. *If $f \in N^{1,p}(\overline{\Omega};\mu_0)$, then $\tilde{f} := f \circ \Phi \in N^{1,p}(\overline{\Omega}^M)$. Moreover,*

$$\|\tilde{f}\|_{N^{1,p}(\overline{\Omega}^M)} = \|f\|_{N^{1,p}(\Omega)} = \|f\|_{N^{1,p}(\overline{\Omega};\mu_0)}.$$

Proof. Let $g \in L^p(\overline{\Omega})$ be an upper gradient of f on $\overline{\Omega}$ and let $\tilde{g} := g \circ \Phi$. We shall show that $\tilde{g}$ is an upper gradient of $\tilde{f}$ on $\overline{\Omega}^M$. To this end, let $\widetilde{\gamma} : [0, l_{\widetilde{\gamma}}] \to \overline{\Omega}^M$ be a curve, parameterized by arc length, and $\gamma = \Phi \circ \widetilde{\gamma}$. As a composition of two 1-Lipschitz functions, γ is 1-Lipschitz and it follows that it is a rectifiable curve, though it need not be parameterized by arc length. Furthermore,

$$\operatorname{Lip} \gamma(t) := \limsup_{h \to 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|} \le 1.$$

Hence

$$\begin{aligned} |\tilde{f}(\widetilde{\gamma}(0)) - \tilde{f}(\widetilde{\gamma}(l_{\widetilde{\gamma}}))| &= |f(\gamma(0)) - f(\gamma(l_{\widetilde{\gamma}}))| \le \int_\gamma g\,ds \\ &= \int_0^{l_{\widetilde{\gamma}}} g \circ \gamma(t) \operatorname{Lip} \gamma(t)\,dt \le \int_0^{l_{\widetilde{\gamma}}} \tilde{g} \circ \widetilde{\gamma}(t)\,dt = \int_{\widetilde{\gamma}} \tilde{g}\,ds, \end{aligned}$$

and so $\tilde{g} \in L^p(\overline{\Omega}^M)$ is an upper gradient of $\tilde{f}$. Hence $\tilde{f} \in N^{1,p}(\overline{\Omega}^M)$, since

$$\|\tilde{f}\|_{L^p(\overline{\Omega}^M)} = \|f\|_{L^p(\Omega)} = \|f\|_{L^p(\overline{\Omega};\mu_0)}.$$

Here we have used the fact that $\mu(\partial_M \Omega) = \mu_0(\partial\Omega) = 0$.

The restriction to Ω of the minimal p-weak upper gradient g_f of f on $\overline{\Omega}$ is minimal also as a p-weak upper gradient on Ω, i.e. if we denote the minimal p-weak upper gradient of $f|_\Omega$ by $g_{f|_\Omega}$, then $g_f|_\Omega = g_{f|_\Omega}$ in Ω. This follows from Shanmugalingam [46, Lemma 3.2], see also Björn–Björn [12, Lemma 2.23]. Similarly $g_{\tilde{f}}|_\Omega = g_{\tilde{f}|_\Omega} = g_{f|_\Omega}$ in Ω. Hence

$$\|g_{\tilde{f}}\|_{L^p(\overline{\Omega}^M)} = \|g_{f|_\Omega}\|_{L^p(\Omega)} = \|g_f\|_{L^p(\overline{\Omega};\mu_0)}. \qquad \square$$

Lemma 5.2. *Let $E \subset \overline{\Omega}$. Then*

$$\overline{C}_p(\Phi^{-1}(E);\Omega^M) \le \overline{C}_p(E;\Omega) \le C_p(E).$$

The first inequality is actually an equality, see Proposition A.5. Further comparisons of these capacities will be given in the appendix.

Proof. For the first inequality, take $u \in \mathcal{A}_E$ with respect to $N^{1,p}(\Omega)$ and let $\tilde{u} = u \circ \Phi : \Omega^M \to \overline{\mathbf{R}}$. Let $x \in \Phi^{-1}(E) \cap \partial_M \Omega$ and take a sequence $y_j \in \Omega^M$ such that $y_j \xrightarrow{d_M} x$. Since $d \le d_M$, $y_j \to \Phi(x)$ in the given metric d. Hence, because $\Phi(x) \in E \cap \partial\Omega$,

$$\liminf_{j \to \infty} u(y_j) \ge 1.$$

Thus $\tilde{u}$ is admissible in the definition of $\overline{C}_p(\Phi^{-1}(E);\Omega^M)$. Taking infimum over all such u, together with the fact that $N^{1,p}(\Omega) = N^{1,p}(\Omega^M)$ (with the same norms), shows the first inequality.

We now turn to the second inequality, which is obvious if $C_p(E) = \infty$. Thus we may assume that $C_p(E) < \infty$. Let $\varepsilon > 0$. Since C_p is an outer capacity, by Theorem 2.5 (a) we can find an open set $G \supset E$ such that $C_p(G) < C_p(E) + \varepsilon$. Therefore we can find $u \in N^{1,p}(X)$ such that $u \geq 1$ in G and $\|u\|^p_{N^{1,p}(X)} < C_p(E)+\varepsilon$. Since G is open, we have $u \in \mathcal{A}_E$. Hence

$$\overline{C}_p(E;\Omega) \leq \|u\|^p_{N^{1,p}(\Omega)} \leq C_p(E) + \varepsilon.$$

Letting $\varepsilon \to 0$ concludes the proof. $\square$

To discuss solutions of Dirichlet problems, we need to compare boundary values of Newtonian functions. Two Newtonian functions have the same boundary values if their difference belongs to $N^{1,p}_0(\Omega) := N^{1,p}_0(\Omega;X) = N^{1,p}_0(\Omega;\overline{\Omega})$, where the last equality was pointed out in Section 2. Similarly, we can define $N^{1,p}_0(\Omega^M) := N^{1,p}_0(\Omega^M;\overline{\Omega}^M)$ and $N^{1,p}_0(\Omega;\mu_0) := N^{1,p}_0(\Omega;(\overline{\Omega},\mu_0))$. The capacity $\overline{C}_p(\,\cdot\,;\Omega^M)$, and the related quasicontinuity, are defined in a manner similar to $\overline{C}_p(\,\cdot\,;\Omega)$, and all the results in Section 3 have direct counterparts for $\overline{C}_p(\,\cdot\,;\Omega^M)$.

The following is the main result of this section. Note that in general, as sets,

$$N^{1,p}(\overline{\Omega}) \subsetneq N^{1,p}(\overline{\Omega};\mu_0) \neq N^{1,p}(\overline{\Omega}^M),$$

and so it is the requirement that $u = 0$ outside Ω which gives the following equality.

Proposition 5.3. *We have $N^{1,p}_0(\Omega) = N^{1,p}_0(\Omega;\mu_0) = N^{1,p}_0(\Omega^M)$.*

Furthermore, if f is a function in this class, then g is a (p-weak) upper gradient of the zero-extension of f with respect to $\overline{\Omega}$ if and only if $g \circ \Phi$ is a (p-weak) upper gradient of f with respect to $\overline{\Omega}^M$. Here Φ is the extension of the canonical identity map $\Omega \to \Omega^M$ to their respective closures.

Proof. The inclusion $N^{1,p}_0(\Omega) \subset N^{1,p}_0(\Omega;\mu_0)$ is clear.

Assume next that $f \in N^{1,p}_0(\Omega;\mu_0)$. Then by definition $f \in N^{1,p}(\overline{\Omega};\mu_0)$ with $f = 0$ on $\partial\Omega$. Lemma 5.1 yields $f \circ \Phi \in N^{1,p}(\overline{\Omega}^M)$, and as $f \circ \Phi \equiv 0$ on $\partial_M\Omega$ we see that $f \circ \Phi \in N^{1,p}_0(\Omega^M)$. Since $f \equiv f \circ \Phi$ on $\Omega = \Omega^M$, we conclude that $N^{1,p}_0(\Omega;\mu_0) \subset N^{1,p}_0(\Omega^M)$.

Finally, if $f \in N^{1,p}_0(\Omega^M)$, then $f \in N^{1,p}(\overline{\Omega}^M)$ with $f = 0$ on $\partial_M\Omega$. Let $g \in L^p(\overline{\Omega}^M)$ be an upper gradient of f on $\overline{\Omega}^M$, and set

$$\tilde{f} = \begin{cases} f & \text{in } \Omega, \\ 0 & \text{on } X \setminus \Omega, \end{cases} \qquad \text{and} \qquad \tilde{g} = \begin{cases} g & \text{in } \Omega, \\ 0 & \text{on } X \setminus \Omega. \end{cases}$$

We shall show that $\tilde{g}$ is an upper gradient of $\tilde{f}$ on X. Let $\gamma : [0, l_\gamma] \to X$ be a rectifiable curve. If $\gamma([0,l_\gamma]) \subset \Omega$, then clearly

$$|\tilde{f}(\gamma(0)) - \tilde{f}(\gamma(l_\gamma))| = |f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g\,ds = \int_\gamma \tilde{g}\,ds,$$

as arc lengths are the same with respect to the metrics d and d_M. If $\gamma([0,l_\gamma]) \cap \Omega$ is empty, then the above inequality is trivial. We may therefore (by splitting γ into two parts and reversing the orientation if necessary) assume that $\gamma([0,l_\gamma]) \not\subset \Omega$ and that $\gamma(0) \in \Omega$. Let $t_0 = \inf\{0 \leq t \leq l_\gamma : \gamma(t) \notin \Omega\} > 0$ and $t_1 = \sup\{0 \leq t \leq l_\gamma :$

$\gamma(t) \notin \Omega\} \le l_\gamma$. Because Ω is open, we have $\gamma(t_0), \gamma(t_1) \in X \setminus \Omega$. We want to show that

$$|\tilde{f}(\gamma(0)) - \tilde{f}(\gamma(t_0))| \le \int_{\gamma|_{[0,t_0)}} \tilde{g}\, ds.$$

Defining the map $\widetilde{\gamma} : [0, t_0) \to \Omega^M$ by $\widetilde{\gamma}(t) = \gamma(t)$ for $0 \le t < t_0$, we note that γ is arc-length parameterized with respect to d_M, by Lemma 4.4. It follows that $\widetilde{\gamma}$ is a 1-Lipschitz map from $[0, t_0)$ to Ω^M, and hence has a continuous extension, also denoted $\widetilde{\gamma}$, from $[0, t_0]$ to $\overline{\Omega}^M$. If $\widetilde{\gamma}(t_0) \in \Omega^M = \Omega$, then so would $\gamma(t_0)$. Thus we conclude that $\widetilde{\gamma}(t_0) \in \partial_M \Omega$, i.e. $\tilde{f}(\gamma(t_0)) = f(\widetilde{\gamma}(t_0)) = 0$. Hence

$$|\tilde{f}(\gamma(0)) - \tilde{f}(\gamma(t_0))| = |f(\widetilde{\gamma}(0)) - f(\widetilde{\gamma}(t_0))| \le \int_{\widetilde{\gamma}|_{[0,t_0)}} g\, ds = \int_{\gamma|_{[0,t_0)}} \tilde{g}\, ds,$$

as g is an upper gradient of f on $\overline{\Omega}^M$. Similarly, if $t_1 < l_\gamma$, then

$$|\tilde{f}(\gamma(t_1)) - \tilde{f}(\gamma(l_\gamma))| \le \int_{\gamma|_{(t_1,l_\gamma]}} \tilde{g}\, ds.$$

The above inequality holds trivially if $t_1 = l_\gamma$. Hence

$$\begin{aligned} |\tilde{f}(\gamma(0)) - \tilde{f}(\gamma(l_\gamma))| &\le |\tilde{f}(\gamma(0)) - \tilde{f}(\gamma(t_0))| + |\tilde{f}(\gamma(t_0)) - \tilde{f}(\gamma(t_1))| \\ &\quad + |\tilde{f}(\gamma(t_1)) - \tilde{f}(\gamma(l_\gamma))| \\ &\le \int_{\gamma|_{[0,t_0)}} \tilde{g}\, ds + 0 + \int_{\gamma|_{(t_1,l_\gamma]}} \tilde{g}\, ds \\ &\le \int_\gamma \tilde{g}\, ds. \end{aligned}$$

Thus $\tilde{g} \in L^p(X)$ is an upper gradient of f on X, and hence $f \in N_0^{1,p}(\Omega)$. □

For the Dirichlet problem in this paper we will also need the following consequence of Proposition 5.3.

Proposition 5.4. *Let $f \in N_0^{1,p}(\Omega^M)$. Then f, extended by 0 to $\partial_M \Omega$, is $\overline{C}_p(\,\cdot\,; \Omega^M)$-quasicontinuous.*

Proof. Let

$$\tilde{f} = \begin{cases} f & \text{in } \Omega, \\ 0 & \text{on } X \setminus \Omega. \end{cases}$$

By Proposition 5.3, $\tilde{f} \in N_0^{1,p}(\Omega)$. So $\tilde{f} \in N^{1,p}(X)$, and hence by Theorem 2.5 (b) it is quasicontinuous in X. For $\varepsilon > 0$ there exists an open set $U \subset X$ with $C_p(U) < \varepsilon$ such that $\tilde{f}|_{X \setminus U}$ is continuous. Then $\widetilde{U} := \Phi^{-1}(U)$ is open in $\overline{\Omega}^M$ by the continuity of Φ. Moreover, $f = \tilde{f} \circ \Phi$ and therefore

$$f|_{\overline{\Omega}^M \setminus \widetilde{U}} = \tilde{f}|_{X \setminus U} \circ \Phi$$

is continuous. Lemma 5.2 shows that $\overline{C}_p(\widetilde{U}; \Omega^M) \le C_p(U) < \varepsilon$. As $\varepsilon > 0$ was arbitrary this shows that f is $\overline{C}_p(\,\cdot\,; \Omega^M)$-quasicontinuous in $\overline{\Omega}^M$. □

6. p-harmonic and superharmonic functions

In this section we introduce p-harmonic and superharmonic functions, as well as obstacle problems, which all will be needed in later sections. For further discussion and references on these topics see Björn–Björn [12] (which also contains proofs of the facts mentioned in this section).

Definition 6.1. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *(super)minimizer* in Ω if

$$\int_{\varphi \neq 0} g_u^p \, d\mu \le \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu \quad \text{for all (nonnegative) } \varphi \in N^{1,p}_0(\Omega).$$

A *p-harmonic function* is a continuous minimizer.

For characterizations of minimizers and superminimizers see A. Björn [7]. Minimizers were first studied for functions in $N^{1,p}(X)$ in Shanmugalingam [46], and it was shown in Kinnunen–Shanmugalingam [33] that under the standing assumptions of this paper, minimizers can be modified on a set of zero capacity to obtain a p-harmonic function. For a superminimizer u, it was shown by Kinnunen–Martio [32] that its *lower semicontinuous regularization*

$$u^*(x) := \operatorname*{ess\,lim\,inf}_{y \to x} u(y) = \lim_{r \to 0} \operatorname*{ess\,inf}_{B(x,r)} u \tag{6.1}$$

is also a superminimizer and $u^* = u$ q.e.

We follow Kinnunen–Martio [32] in the following definition of the obstacle problem. Let $V \subset X$ be a nonempty bounded open set with $C_p(X \setminus V) > 0$. (If X is unbounded then the condition $C_p(X \setminus V) > 0$ is of course immediately fulfilled.)

Definition 6.2. For $f \in N^{1,p}(V)$ and $\psi : V \to \overline{\mathbf{R}}$, we set

$$\mathcal{K}_{\psi,f}(V) = \{v \in N^{1,p}(V) : v - f \in N^{1,p}_0(V) \text{ and } v \ge \psi \text{ a.e. in } V\}.$$

A function $u \in \mathcal{K}_{\psi,f}(V)$ is a *solution of the $\mathcal{K}_{\psi,f}(V)$-obstacle problem* if

$$\int_V g_u^p \, d\mu \le \int_V g_v^p \, d\mu \quad \text{for all } v \in \mathcal{K}_{\psi,f}(V).$$

A solution to the $\mathcal{K}_{\psi,f}(V)$-obstacle problem is easily seen to be a superminimizer in V. Conversely, a superminimizer u in Ω is a solution of the $\mathcal{K}_{u,u}(V)$-obstacle problem for all $V \Subset \Omega$, i.e. V such that $\overline{V}$ is a compact subset of Ω.

Kinnunen–Martio [32, Theorem 3.2] showed that if $\mathcal{K}_{\psi,f}(V)$ is nonempty, then there is a solution u of the $\mathcal{K}_{\psi,f}(V)$-obstacle problem, and this solution is unique up to equivalence in $N^{1,p}(V)$. Moreover, u^* is the unique lower semicontinuously regularized solution. If the obstacle ψ is continuous, then u^* is also continuous, see [32, Theorem 5.5]. The obstacle ψ, as a continuous function, is even allowed to take the value $-\infty$. Given $f \in N^{1,p}(V)$, we let $H_V f$ denote the continuous solution of the $\mathcal{K}_{-\infty,f}(V)$-obstacle problem; this function is p-harmonic in V and takes on the same boundary values (in the Sobolev sense) as f on ∂V, and hence it is also called the solution of the Dirichlet problem with Sobolev boundary values. When $f \in N^{1,p}(X)$ this solution agrees with the one constructed in [46] and studied in [33].

Definition 6.3. A function $u : \Omega \to (-\infty, \infty]$ is *superharmonic* in Ω if

(i) u is lower semicontinuous;
(ii) u is not identically ∞ in any component of Ω;
(iii) for every nonempty open set $V \Subset \Omega$ and all functions $v \in \operatorname{Lip}(X)$, we have $H_V v \le u$ in V whenever $v \le u$ on ∂V.

A function $u : \Omega \to [-\infty, \infty)$ is *subharmonic* in Ω if $-u$ is superharmonic.

This definition of superharmonicity is equivalent to the ones in Heinonen–Kilpeläinen–Martio [25] and Kinnunen–Martio [32], see Theorem 6.1 in A. Björn [6]. A locally bounded superharmonic function is a superminimizer, and all superharmonic functions are lower semicontinuously regularized. Conversely, any lower semicontinuously regularized superminimizer is superharmonic.

By Proposition 5.3, functions that are p-harmonic on an open subset of Ω with respect to either of the metrics d and d_M will be p-harmonic on that subset with respect to both metrics. By Proposition 5.3 we also see that if $f \in N^{1,p}(\Omega) = N^{1,p}(\Omega^M)$, then $\mathcal{K}_{\psi,f}(\Omega) = \mathcal{K}_{\psi,f}(\Omega^M)$, and thus the obstacle problem is exactly the same for both metrics. In particular, $H_\Omega f = H_{\Omega^M} f$ for $f \in N^{1,p}(\Omega)$.

If we let $V \Subset \Omega$ and equip it with the Mazurkiewicz distance d_M and the measure μ, both inherited from Ω, we similarly see that $\mathcal{K}_{\psi,f}(V) = \mathcal{K}_{\psi,f}(V; d_M)$ since the metrics are equivalent on $\overline{V}$ and arc lengths are the same. It follows that also the class of all superharmonic functions on Ω is the same with respect to both metrics d and d_M. Thus within Ω we have no reason to distinguish between, e.g., p-harmonic functions defined using the metric d and the metric d_M.

7. Perron solutions with respect to Ω^M

In addition to the standing assumptions described at the end of Section 3, we assume in this section that Ω is a bounded domain which is finitely connected at the boundary and that, as in Section 6, $C_p(X \setminus \Omega) > 0$.

The main point of this paper is that in considering the Dirichlet boundary value problem there is a difference between Ω and Ω^M. However, we saw in the previous section that the Sobolev solutions $H_\Omega f$ and $H_{\Omega^M} f$ coincide for $f \in N^{1,p}(\Omega)$. For this reason we will usually denote this common solution by Hf. We also write $\mathcal{K}_{\psi,f} := \mathcal{K}_{\psi,f}(\Omega)$.

We shall now consider the Dirichlet problem for arbitrary functions defined on the Mazurkiewicz boundary $\partial_M \Omega$. This will be done by means of Perron solutions on Ω^M defined below. The distinction from Perron solutions on Ω is subtle but has important consequences for the Dirichlet problem since the Mazurkiewicz boundary $\partial_M \Omega$ is finer than $\partial \Omega$.

Definition 7.1. Given a function $f : \partial_M \Omega \to \overline{\mathbf{R}}$, let $\mathcal{U}_f(\Omega^M)$ be the set of all superharmonic functions u on Ω^M, bounded from below, such that

$$\liminf_{\Omega \ni y \xrightarrow{d_M} x} u(y) \geq f(x) \quad \text{for all } x \in \partial_M \Omega.$$

The *upper Perron solution* of f is the function

$$\overline{P}_{\Omega^M} f(x) = \inf_{u \in \mathcal{U}_f(\Omega^M)} u(x), \quad x \in \Omega.$$

Similarly, let $\mathcal{L}_f(\Omega^M)$ be the set of all subharmonic functions u on Ω, bounded from above, such that

$$\limsup_{\Omega \ni y \xrightarrow{d_M} x} u(y) \leq f(x) \quad \text{for all } x \in \partial_M \Omega,$$

and define the *lower Perron solution* of f by

$$\underline{P}_{\Omega^M} f(x) = \sup_{u \in \mathcal{L}_f(\Omega^M)} u(x), \quad x \in \Omega.$$

If $\overline{P}_{\Omega^M} f = \underline{P}_{\Omega^M} f$, then we let $P_{\Omega^M} f := \overline{P}_{\Omega^M} f$ and f is said to be *resolutive* with respect to Ω^M.

We similarly define $\overline{P}_\Omega f$, $\underline{P}_\Omega f$ and $P_\Omega f$ for $f : \partial\Omega \to \overline{\mathbf{R}}$.

Immediate consequences of the above definition are that $\underline{P}_{\Omega^M} f = -\overline{P}_{\Omega^M}(-f)$ and that

$$\overline{P}_{\Omega^M} f_1 \le \overline{P}_{\Omega^M} f_2, \quad \text{if } f_1 \le f_2. \tag{7.1}$$

Observe that $\overline{P}_\Omega f$ is p-harmonic unless it is identically $\pm\infty$, see Theorem 4.1 in Björn–Björn–Shanmugalingam [14]. The proof therein applies also to $\overline{P}_{\Omega^M} f$ without any change. The following comparison principle makes it possible to compare the upper and lower Perron solutions.

Proposition 7.2. *Assume that u is superharmonic and that v is subharmonic in Ω. If*

$$\infty \ne \limsup_{\Omega \ni y \xrightarrow{d_M} x} v(y) \le \liminf_{\Omega \ni y \xrightarrow{d_M} x} u(y) \ne -\infty \quad \textit{for all } x \in \partial_M\Omega, \tag{7.2}$$

then $v \le u$ in Ω.

Corollary 7.3. *If $f : \partial_M\Omega \to \mathbf{R}$, then*

$$\overline{P}_{\Omega^M} f \ge \underline{P}_{\Omega^M} f.$$

The result corresponding to Proposition 7.2 with respect to the given metric d was obtained in Kinnunen–Martio [32, Theorem 7.2].

Proof of Proposition 7.2. Let $\Omega_1 \Subset \Omega_2 \Subset \dots \Subset \Omega = \bigcup_{k=1}^\infty \Omega_k$ and $\varepsilon > 0$. For every $x \in \partial_M\Omega$, it follows from (7.2) that

$$\liminf_{\Omega \ni y \xrightarrow{d_M} x} (u(y) - v(y)) \ge 0$$

and hence there is a ball $B_x^M \ni x$ (with respect to the metric d_M on $\overline{\Omega}^M$) such that

$$u - v > -\varepsilon \quad \text{in } B_x^M \cap \Omega.$$

By the compactness of $\overline{\Omega}^M$ (recall that we assume Ω to be finitely connected at the boundary), there are finitely many balls $B_{x_1}^M, \dots, B_{x_N}^M$ and some $k > 1/\varepsilon$ such that

$$\overline{\Omega}^M \subset \Omega_k \cup B_{x_1}^M \cup \dots \cup B_{x_N}^M.$$

It follows that $v < u + \varepsilon$ on $\partial\Omega_k$. An application of [32, Theorem 7.2] to $u + \varepsilon$ and v in Ω_k now tells us that $v \le u + \varepsilon$ on Ω_k. Letting $\varepsilon \to 0$ completes the proof. $\square$

The following is the main result of this section. Its consequences will be given in Section 8.

Theorem 7.4. *Let $f : \overline{\Omega}^M \to \overline{\mathbf{R}}$ be a $\overline{C}_p(\,\cdot\,;\Omega^M)$-quasicontinuous function such that $f|_\Omega \in N^{1,p}(\Omega)$. Then f is resolutive with respect to Ω^M and $P_{\Omega^M} f = Hf$.*

The corresponding result for Ω under the assumption that $f \in N^{1,p}(X)$ (in which case the quasicontinuity of f is automatic by Theorem 2.5 (b)) was obtained in Björn–Björn–Shanmugalingam [14, Theorem 5.1]. The proof here is more intricate since we need to be more careful with issues of quasicontinuity with respect to the capacity $\overline{C}_p$, which does not come for free. We also need the following modification of Lemma 5.3 in Björn–Björn–Shanmugalingam [14].

Lemma 7.5. *Let $\{U_k\}_{k=1}^\infty$ be a decreasing sequence of relatively open sets in $\overline{\Omega}^M$ such that $\overline{C}_p(U_k;\Omega^M) < 2^{-kp}$. Then there exists a decreasing sequence of nonnegative functions $\{\psi_j\}_{j=1}^\infty$ on Ω such that $\|\psi_j\|_{N^{1,p}(\Omega)} < 2^{-j}$ and $\psi_j \ge k-j$ in $U_k \cap \Omega$.*

Proof. Let $\psi_j = \sum_{k=j+1}^\infty f_k$, where $f_k \in \mathcal{A}_{U_k}$ with $\|f_k\|_{N^{1,p}(\Omega)} < 2^{-k}$ are admissible in the definition of $\overline{C}_p(U_k;\Omega^M)$. $\square$

To prove Theorem 7.4 we will also need the following proposition, which summarizes some useful convergence results for obstacle and Dirichlet problems. It consists of special cases of Farnana [20, Theorem 3.3] and Kinnunen–Marola–Martio [31, Theorem 3], but can also be found in Björn–Björn [12] as Proposition 10.18 and Corollary 10.20. For $f, f_j \in N^{1,p}(X)$ these results are due to Kinnunen–Shanmugalingam [34] and Shanmugalingam [47].

Proposition 7.6. *Let $\{f_j\}_{j=1}^\infty$ be a q.e. decreasing sequence of functions in $N^{1,p}(\Omega)$ such that $f_j \to f$ in $N^{1,p}(\Omega)$ as $j\to\infty$. Then Hf_j decreases to Hf locally uniformly in Ω.*

Moreover, if u and u_j are solutions of the $\mathcal{K}_{f,f}$- and $\mathcal{K}_{f_j,f_j}$-obstacle problems, $j=1,2,\dots$, then $\{u_j\}_{j=1}^\infty$ decreases q.e. in Ω to u.

Proof of Theorem 7.4. Assume first that $f \ge 0$. Extend Hf to $\overline{\Omega}^M$ by letting $Hf := f$ on $\partial_M\Omega$. We first show that Hf is $\overline{C}_p(\,\cdot\,;\Omega^M)$-quasicontinuous. Let $h = f - Hf \in N_0^{1,p}(\Omega^M)$, with $h\equiv 0$ on $\partial_M\Omega$. Proposition 5.4 shows that h is $\overline{C}_p(\,\cdot\,;\Omega^M)$-quasicontinuous. Thus, $Hf = f+h$ is also $\overline{C}_p(\,\cdot\,;\Omega^M)$-quasicontinuous on $\overline{\Omega}^M$, by the subadditivity of the $\overline{C}_p(\,\cdot\,;\Omega^M)$-capacity, see Proposition 3.2. Hence we can find relatively open subsets $G_j \subset \overline{\Omega}^M$, $j=1,2,\dots$, such that $\overline{C}_p(G_j;\Omega^M) < 2^{-jp}$ and such that $Hf|_{\overline{\Omega}^M\setminus G_j}$ is continuous. Let $U_k = \bigcup_{j=k+1}^\infty G_j$, $k=1,2,\dots$. Then $\{U_k\}_{k=1}^\infty$ is a decreasing sequence of relatively open subsets of $\overline{\Omega}^M$ such that $\overline{C}_p(U_k;\Omega^M) < 2^{-kp}$ and $Hf|_{\overline{\Omega}^M\setminus U_k}$ is continuous.

Consider the decreasing sequence of nonnegative functions $\{\psi_j\}_{j=1}^\infty$ given by Lemma 7.5 with respect to this sequence of sets. Let $f_j = Hf + \psi_j$ (which is only defined in Ω) and let φ_j be the lower semicontinuously regularized solution of the $\mathcal{K}_{f_j,f_j}$-obstacle problem.

For positive intergers m, by Lemma 7.5,

$$f_j \ge \psi_j \ge m \quad \text{on } U_{m+j}\cap\Omega. \tag{7.3}$$

Let $\varepsilon>0$ and $x\in\partial_M\Omega$. If $x\notin U_{m+j}$, then by the continuity of $Hf|_{\overline{\Omega}^M\setminus U_{m+j}}$ there is a neighbourhood V_x of x in $\overline{\Omega}^M$ such that

$$f_j(y) \ge Hf(y) \ge Hf(x) - \varepsilon = f(x) - \varepsilon \quad \text{for } y \in (V_x\cap\Omega)\setminus U_{m+j}. \tag{7.4}$$

Combining (7.3) and (7.4) we see that for $x\in\partial_M\Omega\setminus U_{m+j}$,

$$f_j \ge \min\{f(x)-\varepsilon, m\} \quad \text{in } V_x\cap\Omega. \tag{7.5}$$

On the other hand, if $x\in U_{m+j}$, then setting $V_x = U_{m+j}$, we see by (7.3) that (7.5) holds as well. As a solution to the $\mathcal{K}_{f_j,f_j}$-obstacle problem, φ_j is lower semicontinuously regularized and $\varphi_j \ge f_j$ q.e. It follows that $\varphi_j(y) \ge \min\{f(x)-\varepsilon,m\}$ for *every* $y\in V_x\cap\Omega$. Hence

$$\liminf_{\Omega\ni y \xrightarrow{d_M} x} \varphi_j(y) \ge \min\{f(x)-\varepsilon, m\}.$$

Letting $\varepsilon \to 0$ and $m \to \infty$, we see that

$$\liminf_{\Omega \ni y \xrightarrow{d_M} x} \varphi_j(y) \geq f(x) \quad \text{for all } x \in \partial_M \Omega.$$

As φ_j is superharmonic, it follows that $\varphi_j \in \mathcal{U}_f(\Omega^M)$, and hence that $\varphi_j \geq \overline{P}_{\Omega^M} f$.

Since Hf clearly is a solution of the $\mathcal{K}_{Hf,Hf}$-obstacle problem, we see by Proposition 7.6 that $\{\varphi_j\}_{j=1}^\infty$ decreases q.e. to Hf. Hence $\overline{P}_{\Omega^M} f \leq Hf$ q.e. in Ω.

Next, let $f \in N^{1,p}(\Omega)$ be arbitrary. Then by (7.1), Proposition 7.6, and the above argument,

$$\overline{P}_{\Omega^M} f \leq \lim_{m \to -\infty} \overline{P}_{\Omega^M} \max\{f, m\} \leq \lim_{m \to -\infty} H \max\{f, m\} = Hf \quad \text{q.e. in } \Omega.$$

Since both $\overline{P}_{\Omega^M} f$ and Hf are continuous, we have $\overline{P}_{\Omega^M} f \leq Hf$ everywhere in Ω. It then follows from Corollary 7.3 that

$$\underline{P}_{\Omega^M} f = -\overline{P}_{\Omega^M}(-f) \geq -H(-f) = Hf \geq \overline{P}_{\Omega^M} f \geq \underline{P}_{\Omega^M} f,$$

and hence that $Hf = \underline{P}_{\Omega^M} f = \overline{P}_{\Omega^M} f$. □

Remark 7.7. It is shown in Adamowicz–Björn–Björn–Shanmugalingam [1] that if Ω is finitely connected at the boundary, then $\overline{\Omega}^M$ is homeomorphic to the prime end closure $\overline{\Omega}^P$ of Ω, using the definition of prime ends therein. Since in this section, and the next, we only use the topology on $\overline{\Omega}^M$ while the Newtonian spaces (and thus also the involved capacities) are only with respect to $\Omega = \Omega^M$ (and not $\overline{\Omega}$), the results in these two sections can equivalently be formulated in terms of $\overline{\Omega}^P$ when Ω is finitely connected at the boundary.

8. Resolutivity of functions on $\partial_M \Omega$

In addition to the standing assumptions described at the end of Section 3, we assume in this section, as in Section 7, that Ω is a bounded domain which is finitely connected at the boundary and that $C_p(X \setminus \Omega) > 0$.

We now deduce some consequences of Theorem 7.4.

Proposition 8.1. *Assume that $f : \overline{\Omega}^M \to \overline{\mathbf{R}}$ is $\overline{C}_p(\,\cdot\,; \Omega^M)$-quasicontinuous and that $f|_\Omega \in N^{1,p}(\Omega)$. Assume further that $h : \partial_M \Omega \to \overline{\mathbf{R}}$ is zero $\overline{C}_p(\,\cdot\,; \Omega^M)$-q.e., i.e. $\overline{C}_p(\{x \in \partial_M \Omega : h(x) \neq 0\}; \Omega^M) = 0$. Then $f + h$ is resolutive with respect to Ω^M and*

$$P_{\Omega^M}(f + h) = P_{\Omega^M} f.$$

Proof. Extend h to Ω by letting $h = 0$ in Ω. Then h is $\overline{C}_p(\,\cdot\,; \Omega^M)$-quasicontinuous, by (the Ω^M version of) Proposition 3.3. The subadditivity of the $\overline{C}_p(\,\cdot\,; \Omega^M)$-capacity shows that also $f+h$ is $\overline{C}_p(\,\cdot\,; \Omega^M)$-quasicontinuous. Moreover $h \in N^{1,p}(\Omega)$.

Since $f + h = f$ in Ω we have $H(f + h) = Hf$. Theorem 7.4 applied to both f and $f + h$ shows that $f + h$ is resolutive with respect to Ω^M and that

$$P_{\Omega^M}(f + h) = H(f + h) = Hf = P_{\Omega^M} f. \qquad \square$$

Theorem 8.2. *Let $f \in C(\partial_M \Omega)$ and $h : \partial_M \Omega \to \overline{\mathbf{R}}$ be a function which is zero $\overline{C}_p(\,\cdot\,; \Omega^M)$-q.e. on $\partial_M \Omega$. Then f and $f + h$ are resolutive with respect to Ω^M and*

$$P_{\Omega^M}(f + h) = P_{\Omega^M} f.$$

Proof. For each $j = 1, 2, ...$, there is a Lipschitz function $f_j \in \operatorname{Lip}(\partial_M\Omega)$ such that $f - 1/j \le f_j \le f + 1/j$ on $\partial_M\Omega$. We can extend f_j to be a Lipschitz function on $\overline{\Omega}^M$, so $f_j \in \operatorname{Lip}(\overline{\Omega}^M) \subset N^{1,p}(\overline{\Omega}^M)$. It follows directly from Definition 7.1 that $\overline{P}_{\Omega^M} f - 1/j \le \overline{P}_{\Omega^M} f_j \le \overline{P}_{\Omega^M} f + 1/j$, and hence $\overline{P}_{\Omega^M} f_j \to \overline{P}_{\Omega^M} f$ uniformly, as $j \to \infty$. The uniform convergences of $\underline{P}_{\Omega^M} f_j$, $\overline{P}_{\Omega^M}(f_j + h)$ and $\underline{P}_{\Omega^M}(f_j + h)$ are proved in the same way. As $f_j \in N^{1,p}(\Omega)$, we have by Proposition 8.1 that $P_{\Omega^M}(f_j + h) = P_{\Omega^M} f_j$. Letting $j \to \infty$ completes the proof. □

As a consequence of these two results we obtain the following uniqueness result.

Corollary 8.3. *Let either $f : \overline{\Omega}^M \to \mathbf{R}$ be a bounded $\overline{C}_p(\,\cdot\,;\Omega^M)$-quasicontinuous function such that $f|_\Omega \in N^{1,p}(\Omega)$, or $f \in C(\partial_M\Omega)$. Let u be a bounded p-harmonic function in Ω. If there is a set $E \subset \partial_M\Omega$ with $\overline{C}_p(E;\Omega^M) = 0$ such that*

$$\lim_{\Omega \ni y \xrightarrow{d_M} x} u(y) = f(x) \quad \textit{for all } x \in \partial_M\Omega \setminus E,$$

then $u = P_{\Omega^M} f$.

Note that if the word *bounded* is omitted, the result becomes false; consider for example, the Poisson kernel in the unit disc $B(0,1) \subset \mathbf{C} = \mathbf{R}^2$ with a pole at 1 but vanishing on $\partial B(0,1) \setminus \{1\}$.

Proof. By adding a sufficiently large constant to both f and u, and then rescaling them simultaneously we may assume without loss of generality that $0 \le u \le 1$ and $0 \le f \le 1$. Hence $u \in \mathcal{U}_{f-\chi_E}(\Omega^M)$ and $u \in \mathcal{L}_{f+\chi_E}(\Omega^M)$. Therefore, by Proposition 8.1 if f is $\overline{C}_p(\,\cdot\,;\Omega^M)$-quasicontinuous and bounded with $f|_\Omega \in N^{1,p}(\Omega)$, and Theorem 8.2 in the case when $f \in C(\partial_M\Omega)$, we see that

$$u \ge \overline{P}_{\Omega^M}(f - \chi_E) = P_{\Omega^M} f = \underline{P}_{\Omega^M}(f + \chi_E) \ge u. \qquad \square$$

The proofs of the following results are similar to the proof of Theorem 8.2, and are left to the reader to verify.

Proposition 8.4. *Let $f_j : \partial_M\Omega \to \overline{\mathbf{R}}$, $j = 1, 2, \dots$, be resolutive functions with respect to Ω^M and assume that $f_j \to f$ uniformly on $\partial_M\Omega$. Then f is resolutive with respect to Ω^M and $P_{\Omega^M} f_j \to P_{\Omega^M} f$ uniformly in Ω.*

Proposition 8.5. *Let $f_j : \overline{\Omega}^M \to \overline{\mathbf{R}}$ be $\overline{C}_p(\,\cdot\,;\Omega^M)$-quasicontinuous functions such that $f_j|_\Omega \in N^{1,p}(\Omega)$, $j = 1, 2, \dots$. Assume also that $f_j \to f$ uniformly on $\partial_M\Omega$ as $j \to \infty$. Let $h : \partial_M\Omega \to \overline{\mathbf{R}}$ be a function which is zero $\overline{C}_p(\,\cdot\,;\Omega^M)$-q.e. on $\partial_M\Omega$. Then f and $f + h$ are resolutive with respect to Ω^M and $P_{\Omega^M} f = P_{\Omega^M}(f + h)$.*

9. Resolutivity of functions on $\partial\Omega$

In addition to the standing assumptions described at the end of Section 3, we assume in this section that Ω is a nonempty bounded open set (not necessarily finitely connected at the boundary) and that $C_p(X \setminus \Omega) > 0$.

The results in Section 8 have analogs for Perron solutions with respect to the ordinary boundary $\partial\Omega$. Versions of these counterparts appear in Björn–Björn–Shanmugalingam [14] and Björn–Björn [12, Chapter 10] under more restrictive assumptions such as $f \in N^{1,p}(X)$ and $f \in N^{1,p}(\overline{\Omega})$ respectively. The capacities considered there are C_p and $C_p(\,\cdot\,;\overline{\Omega})$. The generalizations below have been made possible by the introduction of the new capacity $\overline{C}_p(\,\cdot\,;\Omega)$ in this paper. For a comparison of these results see the examples in Section 10.

We next list these generalizations without proofs since the verification of these results follow directly along the lines of the proofs of Theorem 7.4 and the results in Section 8. For readers only interested in the results in this section, we point out that some details are easier, for the only result from Sections 4 and 5 needed for the results in this section is Proposition 5.4, which however has a much simpler proof in this case. Moreover, there is no need to assume that Ω is connected or finitely connected at the boundary for the results in this section.

Theorem 9.1. *Let $f:\overline{\Omega}\to\overline{\mathbf{R}}$ be $\overline{C}_p(\,\cdot\,;\Omega)$-quasicontinuous with $f|_\Omega\in N^{1,p}(\Omega)$. Assume that $h:\partial\Omega\to\overline{\mathbf{R}}$ is zero $\overline{C}_p(\,\cdot\,;\Omega)$-q.e. Then f and $f+h$ are resolutive with respect to Ω and*

$$P_\Omega(f+h)=P_\Omega f=Hf.$$

Proposition 9.2. *Let $f\in C(\partial\Omega)$ and let h be a function which is zero $\overline{C}_p(\,\cdot\,;\Omega)$-q.e. on $\partial\Omega$. Then $f+h$ is resolutive with respect to Ω and*

$$P_\Omega(f+h)=P_\Omega f.$$

Note that the resolutivity of $f\in C(\partial\Omega)$ was already obtained in Björn–Björn–Shanmugalingam [14, Theorem 6.1].

Corollary 9.3. *Let either $f:\overline{\Omega}\to\mathbf{R}$ be a bounded $\overline{C}_p(\,\cdot\,;\Omega)$-quasicontinuous function such that $f|_\Omega\in N^{1,p}(\Omega)$, or $f\in C(\partial\Omega)$. Assume also that u is a bounded p-harmonic function in Ω and that there is a set $E\subset\partial\Omega$ with $\overline{C}_p(E;\Omega)=0$ such that*

$$\lim_{\Omega\ni y\to x}u(y)=f(x)\quad\textit{for all } x\in\partial\Omega\setminus E.$$

Then $u=P_\Omega f$.

The following is a convenient existence and uniqueness result for solutions of the Dirichlet problem with continuous boundary data. With $\overline{C}_p(\,\cdot\,;\Omega)$ replaced by $C_p(\,\cdot\,)$ it was probably first given explicitly in Björn–Björn [11] as a consequence of Corollary 6.2 in Björn–Björn–Shanmugalingam [14] and the Kellogg property.

Corollary 9.4. *Assume that $f\in C(\partial\Omega)$. Then there is a unique bounded p-harmonic function u on Ω such that*

$$\lim_{\Omega\ni y\to x}u(y)=f(x)\quad\textit{for } \overline{C}_p(\,\cdot\,;\Omega)\textit{-q.e. } x\in\partial\Omega. \tag{9.1}$$

Moreover, $u=P_\Omega f$.

Proof. We already know by Corollary 9.3 that if u is a bounded p-harmonic function on Ω satisfying (9.1), then $u=P_\Omega f$, which shows the uniqueness.

As for the existence, let $u=P_\Omega f$. That f is resolutive with respect to Ω follows from Proposition 9.2, but was actually first shown in Björn–Björn–Shanmugalingam [14, Theorem 6.1]. An application of Björn–Björn–Shanmugalingam [13, Theorem 3.9] together with [14, Theorem 6.1] shows that there is a set $E\subset\partial\Omega$ such that $C_p(E)=0$ and

$$\lim_{\Omega\ni y\to x}u(y)=f(x)\quad\text{for all } x\in\partial\Omega\setminus E.$$

By Lemma 5.2, $\overline{C}_p(E,\Omega)=0$. ☐

10. Examples and applications

The results in this paper are the third generation of this type of results, following Björn–Björn–Shanmugalingam [14] and Björn–Björn [12]. In this section we give some new examples illustrating the results of this paper (in some cases, in combination with the results found in [12]). These examples are not covered by the results found in [14]. They also demonstrate the differences between the capacities considered in this paper. There are also some resolutivity results in A. Björn [8] which are relevant for our discussion, see Example 10.1.

Example 10.1. (Cusps in $\mathbf{R}^2$) Let $X = \mathbf{R}^2$ (unweighted) and $p > 2$. It is well-known that $C_p(\{x\}) > 0$ for each $x \in \mathbf{R}^2$. Let Ω be the cusp

$$\Omega = \{(x_1, x_2) : 0 < x_1 < 1 \text{ and } 0 < x_2 < x_1^\beta\}$$

with $\beta > p - 1$. By considering the functions $u_R(x_1, x_2) = \max\{1 - x_1/R, 0\}$, $0 < R < 1$, we see that $\overline{C}_p(\{0\}; \Omega) = 0$ and that $C_p(\{0\}; \overline{\Omega}) = 0$.

This means that in Theorem 9.1 and Proposition 9.2 we can change the boundary data arbitrarily at the origin when considering the Perron solution with respect to Ω, even though $C_p(\{0\}) > 0$. Similarly, the exceptional set E in Corollary 9.3 can contain 0. This improves upon the perturbation results of Björn–Björn–Shanmugalingam [14].

Note that any function in $N^{1,p}(\overline{\Omega})$ is continuous apart from possibly at 0. It thus follows from Propositions 2.6 and 3.3 that it is both $C_p(\,\cdot\,; \overline{\Omega})$- and $\overline{C}_p(\,\cdot\,; \Omega)$-quasicontinuous, so that the resolutivity and perturbation results in Theorem 9.1 apply to all functions in $N^{1,p}(\overline{\Omega})$. For functions in $C(\partial\Omega)$, these results follow also from A. Björn [8, Theorem 1.3]. However, the function $f(x) = \sin \log |x| \in N^{1,p}(\overline{\Omega})$ cannot be treated by [8], regardless of the value given to $f(0)$. By Theorem 9.1, it is resolutive with respect to Ω and Pf is independent of $f(0)$.

Note that as Ω is locally connected at the boundary, we have $\overline{\Omega} = \overline{\Omega}^M$, and thus the Ω^M-Perron solutions are the same as the usual Perron solutions in this case. Consider instead

$$\Omega_0 = (0, 2)^2 \setminus \{(x_1, x_2) : 0 < x_1 \le 1 \text{ and } x_2 = x_1^\beta\} \supset \Omega.$$

Then $\overline{\Omega}_0^M \ne \overline{\Omega}_0$ since the origin splits into 0_1 and 0_2 depending on whether it is approached from the right (x_1-direction) or from above (x_2-direction). Then $\overline{C}_p(\{0\}; \Omega_0)$, $C_p(\{0\}; \overline{\Omega}_0)$, $\overline{C}_p(\{0_2\}; \Omega_0^M)$ and $C_p(\{0_2\}; \overline{\Omega}_0^M)$ are all positive and neither 0 nor 0_2 can be treated by the results in Section 9. On the other hand, the discussion in the first paragraph of this example shows that $\overline{C}_p(\{0_1\}; \Omega_0^M) = C_p(\{0_1\}; \overline{\Omega}_0^M) = 0$, so all the above resolutivity and perturbation results apply to $\{0_1\}$. For example, the function

$$f_0(x) = \begin{cases} \sin \log |x|, & \text{if } x \in \overline{\Omega}_0^M,\ |x| \le 1 \text{ and } d_M(x, 0_1) \le d_M(x, 0_2), \\ 0, & \text{otherwise}, \end{cases}$$

is resolutive with respect to Ω_0^M and Pf is independent of $f(0_1)$.

To obtain similar results for $1 < p \le 2$, equip $\mathbf{R}^2$ with the measure $|x|^{-1}\,dx$ (which is doubling and supports a p-Poincaré inequality by Heinonen–Kilpeläinen–Martio [25, Example 2.22]) and let $\beta > p$ in the above construction.

Example 10.2. (The topologist's comb) Let $\Omega \subset \mathbf{R}^2$ be given by

$$\Omega := ((0, 2) \times (-1, 1)) \setminus \left(\left\{1, \tfrac{1}{2}, \tfrac{1}{4}, \dots, 0\right\} \times [0, 1)\right).$$

Let $A = \{0\} \times (0,1] \subset \partial\Omega$. Then $\overline{\Omega} = [0,2] \times [-1,1]$ and so $C_p(A;\overline{\Omega}) > 0$. As we shall see, $\overline{C}_p(A;\Omega) = 0$, and thus we get significantly better results using the $\overline{C}_p(\,\cdot\,;\Omega)$-capacity than with the $C_p(\,\cdot\,;\overline{\Omega})$-capacity. To show that $\overline{C}_p(A;\Omega) = 0$ we let

$$\delta_j = \biggl(\frac{3}{4}\biggr)^{j/(p-1)} \quad \text{and} \quad f_j(x,y) = \begin{cases} \min\biggl\{\dfrac{y}{\delta_j}, 1\biggr\}, & \text{if } 2^{-j} < x < 2^{1-j},\ 0 < y < 1, \\ 0, & \text{otherwise,} \end{cases}$$

for $j = 1, 2 \dots$. Set $h_k := \sum_{j=k}^\infty f_j \in \mathcal{A}_A$. Then $\|h_k\|_{L^p(\Omega)}^p \to 0$ as $k \to \infty$, and

$$\|g_{h_k}\|_{L^p(\Omega)}^p = \sum_{j=k}^\infty \frac{2^{-j}\delta_j}{\delta_j^p} = \sum_{j=k}^\infty \biggl(\frac{2}{3}\biggr)^j \to 0, \quad \text{as } k \to \infty.$$

Hence for all $p > 1$, $\overline{C}_p(A;\Omega) \le \|h_k\|_{N^{1,p}(\Omega)}^p \to 0$, as $k \to \infty$. (Because singleton sets have zero capacity if and only if $1 \le p \le 2$, it also follows that $\overline{C}_p(\overline{A}, \Omega) = 0$ if and only if $1 < p \le 2$.)

We can therefore perturb the boundary data as we wish on A in Theorem 9.1 and Proposition 9.2, and in Corollary 9.3. In particular, Proposition 9.2 shows that if $f \in C(\partial\Omega)$ and $h = f$ on $\partial\Omega \setminus A$, then h is resolutive with respect to Ω and $Ph = Pf$. None of this can be inferred from the results of Björn–Björn–Shanmugalingam [14], nor from the results in Björn–Björn [12].

A variant of this example is obtained by replacing each slit $S_j = \{2^{-j}\} \times [0,1)$ by the thin rectangle

$$R_j = [2^{-j} - 2^{-j-2}, 2^{-j} + 2^{-j-2}] \times [0,1), \quad j = 0, 1, \dots,$$

i.e. letting $\Omega' = ((0,2) \times (-1,1)) \setminus \bigcup_{j=0}^\infty R_j$. Since $g = \infty\chi_A$ is an upper gradient of χ_A in $\overline{\Omega}'$, we see that $C_p(A;\overline{\Omega}') \le \|\chi_A\|_{N^{1,p}(\overline{\Omega}')} = 0$. Note that any curve starting in A and ending in $\overline{\Omega}' \setminus A$ must pass through $(0,0)$ first, and therefore intersects A along an interval of positive length. On the other hand, $C_p(\overline{A}, \overline{\Omega}') = 0$ if and only if $1 < p \le 2$.

Thus, the above resolutivity and perturbation conclusions for this "thickened" comb Ω' are obtainable already by the results in [12], where they are formulated using $C_p(\,\cdot\,;\overline{\Omega}')$. However, we now show an interesting phenomenon on Ω' which does not appear in the ordinary comb Ω: For $(x,y) \in \overline{\Omega}'$, let

$$f(x,y) = \begin{cases} y, & \text{if } 0 \le y \le 1 \text{ and } x \in \bigcup_{j=1}^\infty (2^{-2j}, 2^{1-2j}), \\ 0, & \text{otherwise.} \end{cases} \tag{10.1}$$

Then $g \equiv 1$ is an upper gradient of f in $\overline{\Omega}'$ and hence $f \in N^{1,p}(\overline{\Omega}')$. Since $\overline{C}_p(A;\Omega') = C_p(A;\overline{\Omega}') = 0$ and f is continuous at all points but those in A, Proposition 3.3 implies that f is both $\overline{C}_p(\,\cdot\,;\Omega')$- and $C_p(\,\cdot\,;\overline{\Omega}')$-quasicontinuous. It is thus resolutive by Theorem 9.1 (and even by Theorem 10.15 in [12]).

Note that f oscillates near A, has countably many "jumps" on lines parallel to the x-axis with ordinate $0 < y < 1$, and cannot be extended to a Newtonian function on $\mathbf{R}^2$. A similar construction is not possible on the ordinary comb Ω since all the slits $\{2^{-j}\} \times [0,1)$ have positive $\overline{C}_p(\,\cdot\,;\Omega)$-capacity and thus a function with jumps at these slits cannot be $\overline{C}_p(\,\cdot\,;\Omega)$-quasicontinuous on $\overline{\Omega}$. See however Example 10.4 below where a similar construction is done on the countable comb. (The reason why it works there is that the union of the main slits has zero $\overline{C}_p(\,\cdot\,,\Omega)$-capacity, making f $\overline{C}_p(\,\cdot\,;\Omega)$-quasicontinuous on $\overline{\Omega}$.)

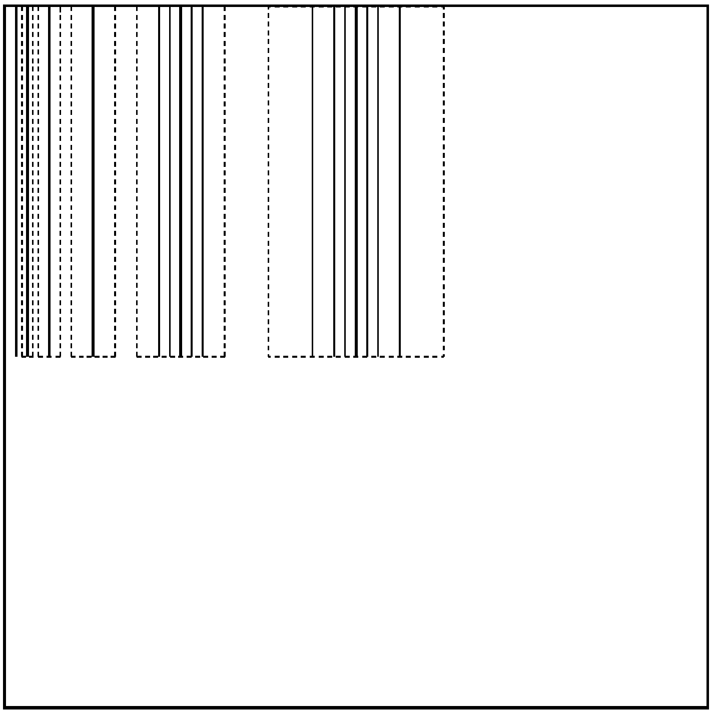

Figure 1. The topologist's combs in Examples 10.2 and 10.4. The thick lines represent the main slits S_j, the broken lines represent the rectangles R_j in the "thickened" comb and the thin lines represent the secondary slits near the first two main slits S_0 and S_1 in the countable comb.

The above disctinction between the ordinary comb and the "thickened" comb further motivates Perron solutions with respect to the Mazurkiewicz boundary $\partial_M\Omega$ and the generalized Perron solutions, see Sections 7, 8 and 11, and also A. Björn [9].

Example 10.3. (Double comb) Let $\Omega \subset \mathbf{R}^2$ be given by

$$\Omega = (-1,1)^2 \setminus \bigl(\bigl\{\pm\tfrac{1}{2}, \pm\tfrac{1}{4}, \pm\tfrac{1}{8}, \dots, 0\bigr\} \times [0,1)\bigr).$$

Just as in Example 10.2 we find that

$$\overline{C}_p(A;\Omega) = 0 < C_p(A;\overline{\Omega}),$$

where $A = \{0\} \times (0,1]$, and we get similar consequences for perturbing the boundary data at points in A as in Example 10.2 (not obtainable from the results in [14] and [12]). Consider e.g. the function

$$f(x,y) = \begin{cases} y, & 0 \le x \le 1,\ 0 < y \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{10.2}$$

Arguing as in Example 10.2, we see that $f \in N^{1,p}(\Omega)$ and that f is $\overline{C}_p(\,\cdot\,,\Omega)$-quasicontinuous. Hence, by Theorem 9.1, f is resolutive even though $f \notin N^{1,p}(\overline{\Omega})$ (since it is not absolutely continuous on lines parallel to the x-axis with ordinate $0 < y < 1$). Therefore, the resolutivity of f *cannot* be obtained by the results in [14] or [12].

Example 10.4. (Countable comb) Let $\Omega \subset \mathbf{R}^2$ be given by

$$\Omega = ((0,2)\times(-1,1)) \setminus \biggl(\biggl(E \cup \bigcup_{j=0}^{\infty} E_j\biggr) \times [0,1)\biggr),$$

where

$$E = \{2^{-j} : j = 0,1,...\} \quad \text{and} \quad E_j = \{2^{-j}(1 \pm 2^{-k}) : k = 3,4,...\}.$$

Furthermore, let $A' = (E \cup \{0\}) \times (0,1]$ and let f be given by (10.1). Note that f oscillates near $\{0\}\times(0,1]$ and has countably many "jumps" on lines parallel to the x-axis with ordinate $0 < y < 1$.

As in Example 10.2 (and using also the countable subadditivity of the capacity) we see that $\overline{C}_p(A',\Omega) = 0$, from which it follows that f is $\overline{C}_p(\,\cdot\,,\Omega)$-quasicontinuous. Moreover, $f \in N^{1,p}(\Omega)$. Hence, f is resolutive by Theorem 9.1.

We next give an example of a domain whose boundary $\partial\Omega \subset \mathbf{R}$ has positive measure, and such that there is a set $K \subset \partial\Omega$ with $\overline{C}_p(K;\Omega) = 0$ and $\mu(\partial\Omega\setminus K) = 0$, i.e. from a measure-theoretic point of view K is essentially all the boundary, but for the Perron solution results in this paper it is negligible.

Example 10.5. Let $c_n = 2^{-n}$, $n = 0,1,\ldots$. We shall construct a Cantor set $\widetilde{K} = \bigcap_{n=0}^{\infty} \widetilde{K}_n \subset [0,2] = \widetilde{K}_0$ inductively as follows. The nth generation $\widetilde{K}_n$ consists of 2^n closed intervals of length $\alpha_n := 2^{-n}(1+c_n) = 2^{-2n}(2^n+1)$, $n = 0,1,\ldots$, and is obtained by removing the open middle subinterval of length

$$\theta_n = \alpha_{n-1} - 2\alpha_n = 2^{1-n}(c_{n-1} - c_n) = 2^{1-n}c_n = 2^{1-2n}$$

from each interval constituting $\widetilde{K}_{n-1}$. It is easy to see that $\widetilde{K}$ has positive length, or more precisely $\Lambda_1(\widetilde{K}) = 1$, where Λ_1 is the 1-dimensional Lebesgue measure.

Set $K_n = \widetilde{K}_n \times \widetilde{K}_n$ and $K = \widetilde{K} \times \widetilde{K}$. The set K_n consists of 4^n closed squares $Q_{n,j}$, $j = 1,\ldots,4^n$, and K has two-dimensional Lebesgue measure $\Lambda_2(K) = 1$. Let

$$x_{n,j} = \inf\{x : (x,y) \in Q_{n,j}\} \quad \text{and} \quad x'_{n,j} = \sup\{x : (x,y) \in Q_{n,j}\} = x_{n,j} + \alpha_n$$

be the left- and right-hand end points of the projection of the square $Q_{n,j}$ to the x-axis, and set

$$A_{n,j,k} = \begin{cases} \{(x,y) : x \geq x_{n,j}\}, & \text{if } k \text{ is even}, \\ \{(x,y) : x \leq x'_{n,j}\}, & \text{if } k \text{ is odd}. \end{cases}$$

Consider the sets

$$F_{n,j,k} = \biggl\{z \in A_{n,j,k} : \operatorname{dist}(z, Q_{n,j}) = \frac{\theta_n}{6}(1 + 2^{-n}k)\biggr\}, \quad k = 0,\ldots,2^n,$$

obtained by the concatenation of three line segments and two quartercircles. Note that all the sets $F_{n,j,k}$ are pairwise disjoint and in fact, $\operatorname{dist}(F_{n,j,k}, F_{n,j,k+1}) = 2^{-n}\theta_n/6$. For $m > n$,

$$\operatorname{dist}(F_{n,j,k}, F_{m,i,l}) \geq \frac{\theta_n}{6} - \frac{\theta_{n+1}}{3} = \frac{2^{-n}(c_n - c_{n+1})}{3} = \frac{\theta_{n+1}}{3} \geq \frac{2^{-n}\theta_n}{6}.$$

Finally we let

$$\Omega = (-1,3)^2 \setminus \biggl(K \cup \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{4^n} \bigcup_{k=1}^{2^n+1} F_{n,j,k}\biggr).$$

Around each square $Q_{n,j}$ we have thus placed $2^n + 1$ pairwise disjoint arcs with openings placed alternately to the left and to the right of the square. This means

that any curve in Ω connecting a point in the region outside of all these arcs with a point in $Q_{n,j}$ must have length at least $2^n\alpha_n = 1 + c_n$. Thus, any curve starting at a point $z \in \Omega$ with $\operatorname{dist}(z, Q_{n,j}) \geq \theta_n/3$ and ending at a point $w \in \Omega$ with $\operatorname{dist}(w, Q_{m,i}) \leq \theta_m/6$, $m > n$, must have length at least $\sum_{k=n}^{m} 2^k\alpha_k \to \infty$ as $m \to \infty$, for each n.

We will now show that $\overline{C}_p(K;\Omega) = 0$. Let

$$f_j(z) = \min\{d_{\text{inner}}(z, (-1,0))/j, 1\}.$$

By the observation above, we see that $f_j \in \mathcal{A}_K$ with $g_{f_j} \leq 1/j$. Moreover, $\|f_j\|^p_{L^p(\Omega)} \to 0$ and $\|g_{f_j}\|^p_{L^p(\Omega)} \leq \mu(\Omega)/j^p \to 0$ as $j \to \infty$. Hence $\overline{C}_p(K;\Omega) \leq \|f_j\|^p_{N^{1,p}(\Omega)} \to 0$ as $j \to \infty$. On the other hand,

$$C_p(K;\overline{\Omega}) = C_p(K;[-1,3]^2) \geq \mu(K) > 0.$$

In fact, we even have $C_p(K;(\overline{\Omega};\mu_0)) > 0$. Indeed, if $C_p(\widetilde{K};[-1,3]\times[-1,0])$ were 0, then a reflection and localization argument would show that $C_p(\widetilde{K};\mathbf{R}^2)$ would be 0, which contradicts Theorem 2.26 in Heinonen–Kilpeläinen–Martio [25] since $\Lambda_1(\widetilde{K}) > 0$. Hence $0 < C_p(\widetilde{K};[-1,3]\times[-1,0]) \leq C_p(K;(\overline{\Omega};\mu_0))$, by monotonicity.

Observe that Ω is boundedly connected at the boundary, and that if $x \in \partial\Omega$, then Ω is either locally connected at x (for $x \in \partial(-1,3)^2 \cup K$) or 2-connected at x (for $x \in F_{n,j,k}$).

The following modification of the example above may be of interest.

Example 10.6. Let Ω be constructed just as in Example 10.5, but replace each $F_{n,j,k}$ by $F'_{n,j,k} = \overline{G}_{n,j,k}$, where $G_{n,j,k}$ is the $2^{-n}\theta_n/24$-neighbourhood of $F_{n,j,k}$. Then all $F'_{n,j,k}$ are still pairwise disjoint and Ω is locally connected at the boundary.

This time, not only $\overline{C}_p(K;\Omega) = 0$ but also $C_p(K;(\overline{\Omega};\mu_0)) = 0$. Indeed, there are no nonconstant rectifiable curves in $\overline{\Omega} = [-1,3]^2 \setminus \bigcup_{n,j,k} G_{n,j,k}$ intersecting K, and hence $\chi_K \in N^{1,p}(\overline{\Omega})$. Note however that $0 < \mu(K) \leq C_p(K;\overline{\Omega}) \leq \|\chi_K\|^p_{N^{1,p}(\overline{\Omega})} = \mu(K)$.

In Examples 10.5 and 10.6 we had $\overline{C}_p(K;\Omega) = 0$ because there were no rectifiable curves in Ω terminating in K. In the following example exery $x \in K$ is accessible by rectifiable curves from Ω but there is still a set $K^* \subset K$ with full measure in K such that $\overline{C}_p(K^*;\Omega) = 0$. Roughly speaking, K^* lies deep in K and there are rather few curves reaching that far.

Example 10.7. Let $\{c_n\}_{n=0}^\infty$ be a strictly decreasing sequence such that $0 < c_0 \leq \frac{1}{3}$ and $\lim_{n\to\infty} c_n = 0$. We shall construct a Cantor set $\widetilde{K} = \bigcap_{n=0}^\infty \widetilde{K}_n \subset [0, 1 + c_0] = \widetilde{K}_0$ inductively as follows. The nth generation $\widetilde{K}_n$ consists of 2^n closed intervals of length $\alpha_n = 2^{-n}(1 + c_n)$, $n = 0, 1, ...$, and is obtained by removing the open middle subinterval of length $\alpha_{n-1} - 2\alpha_n$ from each interval constituting $\widetilde{K}_{n-1}$. It is easy to see that $\widetilde{K}$ has length 1, and in fact

$$\Lambda_1([0,\alpha_n] \cap \widetilde{K}) = \lim_{k\to\infty} 2^{k-n}\alpha_k = \lim_{k\to\infty} 2^{k-n}2^{-k}(1 + c_k) = 2^{-n}, \quad n = 0, 1, \dots, \tag{10.3}$$

where Λ_1 is the 1-dimensional Lebesgue measure.

Set $K_n = \widetilde{K}_n \times \widetilde{K}_n$ and $K = \widetilde{K} \times \widetilde{K}$. The set K_n consists of 4^n closed squares $Q_{n,j}$, $j = 1, \dots, 4^n$, and K has area 1, by (10.3). Let

$$\Omega = (-1,3)^2 \setminus K \subset \mathbf{R}^2.$$

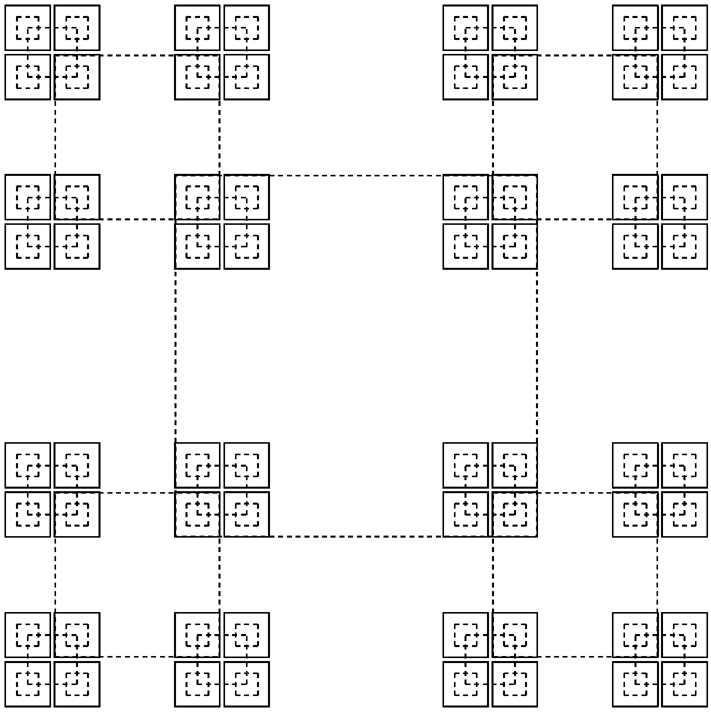

Figure 2. The squares Q^* of the first four generations in the construction in Example 10.7 are drawn by broken lines, and solid lines mark the set K_3.

We shall now construct a set $K^* \subset K$ with area 1 and $\overline{C}_p(K^*;\Omega) = 0$ simultaneously for all $p > 1$. Fix $n \geq 0$ and let $Q = Q_{n,j}$ be one of the 4^n closed squares of sidelength α_n, constituting K_n. Let $Q^* = Q^*_{n,j}$ be the square concentric with Q and of sidelength $\beta_n = \alpha_n - 2\alpha_{n+1} + 2\alpha_{n+2} > 2\alpha_{n+2} > 2^{-n-1}$, see Figure 2. The square Q^* contains four squares of sidelengths α_{n+2} from the set K_{n+2}, each of them belonging to a different component of K_{n+1}.

Let u_Q be a Lipschitz function supported in Q and such that $u_Q = 1$ on Q^* and $|\nabla u_Q| \leq 2/(\alpha_n - \beta_n) = 1/(\alpha_{n+1} - \alpha_{n+2})$, e.g.

$$u_Q(x) = \frac{\max\{1 - \operatorname{dist}(x, Q^*), 0\}}{\alpha_{n+1} - \alpha_{n+2}}.$$

Then

$$\int_\Omega |\nabla u_Q|^p \, d\Lambda_2 \leq \frac{\Lambda_2(Q \cap \Omega)}{(\alpha_{n+1} - \alpha_{n+2})^p}.$$

By translation and (10.3) we see that

$$\Lambda_2(Q \cap \Omega) = \Lambda_2([0, \alpha_n]^2 \setminus K) = \alpha_n^2 - 4^{-n} = 4^{-n}(2c_n + c_n^2) \leq 3 \cdot 4^{-n} c_n. \qquad (10.4)$$

Since $\alpha_{n+1} - \alpha_{n+2} = 2^{-n-2}(1 + 2c_{n+1} - c_{n+2}) > 2^{-n-2}$, this yields

$$\int_\Omega |\nabla u_Q|^p \, d\Lambda_2 \leq 3 \cdot 2^{p(n+2)} 4^{-n} c_n.$$

We also have, using (10.4) again, that

$$\int_\Omega |u_Q|^p \, d\Lambda_2 \leq \Lambda_2(Q \cap \Omega) \leq 3 \cdot 4^{-n} c_n.$$

Next, let

$$K_n^* = \bigcup_{j=1}^{4^n} Q_{n,j}^* \quad \text{and} \quad u_n = \sum_{j=1}^{4^n} u_{Q_{n,j}}, \quad n = 0, 1, \dots .$$

From the last two estimates we conclude that

$$\|u_n\|_{N^{1,p}(\Omega)}^p \le 4^n(3 \cdot 2^{p(n+2)}4^{-n}c_n + 3 \cdot 4^{-n}c_n) \le 4 \cdot 2^{p(n+2)}c_n$$

and hence

$$\|u_n\|_{N^{1,p}(\Omega)} \le 4^{1/p}2^{n+2}c_n^{1/p} \le 16 \cdot 2^n c_n^{1/p}. \tag{10.5}$$

Finally, set

$$K^* = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} K_n^* \quad \text{and} \quad v_k = \sum_{n=k}^{\infty} u_n, \quad k = 0, 1, \dots .$$

As $K_n^* \subset K_n$ for each n, we see that $K^* \subset K$. Moreover, for each k, the function v_k is admissible in the definition of $\overline{C}_p(K^*;\Omega)$ and hence by (10.5),

$$\overline{C}_p(K^*;\Omega)^{1/p} \le \|v_k\|_{N^{1,p}(\Omega)} \le \sum_{n=k}^{\infty} \|u_n\|_{N^{1,p}(\Omega)} \le 16 \sum_{n=k}^{\infty} 2^n c_n^{1/p}.$$

Choosing $c_n = 2^{-n^2}$ we get

$$\overline{C}_p(K^*;\Omega)^{1/p} \le 16 \sum_{n=k}^{\infty} 2^{n-n^2/p}.$$

The sum in the right-hand side converges, since $2^{n-n^2/p} < 2^{-n}$ if $n > 2p$, and thus the right-hand side tends to 0 as $k \to \infty$. Hence $\overline{C}_p(K^*;\Omega) = 0$. (In fact this is true for all $p > 1$ as long as $(\log c_n)/n \to -\infty$ as $n \to \infty$.)

It remains to show that K^* has area 1. Let $k \ge 0$ be fixed and consider the set

$$K \cap \bigcup_{n=k}^{\infty} K_n^*. \tag{10.6}$$

The $(k+2)$-th generation of K is made up of 4^{k+2} parts and $K \cap K_k^*$ consists of exactly 4^{k+1} of these parts. Hence

$$\Lambda_2(K \cap K_k^*) = \frac{4^{k+1}}{4^{k+2}}\Lambda_2(K) = \frac{1}{4}.$$

Similarly, the $(k+3)$-th generation of K is made up of 4^{k+3} parts and $K \cap K_{k+1}^*$ consists of exactly 4^{k+2} of these parts, however one fourth of those are already contained in $K \cap K_k^*$, and thus

$$\Lambda_2(K \cap (K_{k+1}^* \setminus K_k^*)) = \frac{3 \cdot 4^{k+1}}{4^{k+3}}\Lambda_2(K) = \frac{3}{4} \cdot \frac{1}{4}.$$

Proceeding in the same way, we see that

$$\Lambda_2\biggl(K \cap \biggl(K_m^* \setminus \bigcup_{n=k}^{m-1} K_n^*\biggr)\biggr) = \frac{3^{m-k} \cdot 4^{k+1}}{4^{m+2}}\Lambda_2(K) = \left(\frac{3}{4}\right)^{m-k}\frac{1}{4}, \quad m = k, k+1, \dots .$$

Summing up we obtain

$$\Lambda_2\biggl(K \cap \bigcup_{n=k}^{\infty} K_n^*\biggr) = \biggl(1+\frac{3}{4}+\biggl(\frac{3}{4}\biggr)^2+\ldots\biggr)\frac{1}{4} = 1.$$

From which we conclude that

$$\Lambda_2(K^*) = \Lambda_2(K \cap K^*) = \lim_{k\to\infty} \Lambda_2\biggl(K \cap \bigcup_{m=k}^{\infty} K_m^*\biggr) = 1.$$

On the other hand, for $p>1$, we have $\overline{C}_p(K;\Omega) \ge \overline{C}_p(\widetilde{K}\times\{0\};\Omega) > 0$, since every $u \in N^{1,p}(\Omega)$ admissible in the definition of $\overline{C}_p(\widetilde{K}\times\{0\};\Omega)$ gives by reflection in the x-axis rise to $v \in N^{1,p}((-1,3)\times(-1,1))$ with

$$\|v\|^p_{N^{1,p}((-1,3)\times(-1,1))} \le 2\|u\|^p_{N^{1,p}(\Omega)}.$$

It is well known (see e.g. Theorem 2.26 in Heinonen–Kilpeläinen–Martio [25]) that sets of p-capacity zero in $\mathbf{R}^n$ have Hausdorff dimension at most $n-p$ if $p \le n$. (If $n>p$ we instead use that the p-capacity is zero only for the empty set.) Hence as $\Lambda_1(\widetilde{K})>0$, we conclude that $\overline{C}_p(\widetilde{K}\times\{0\};\Omega)>0$.

On the contrary, the set K^* constructed above has positive area (and full measure in K) but zero $\overline{C}_p(\,\cdot\,;\Omega)$-capacity for all $p \ge 1$.

In view of the above examples, the following is a natural question to ask.

Open problem 10.8. Assume that Ω is as in Section 8 and let $E \subset \partial\Omega$ be the set of inaccessible boundary points, see below. Is it then true that

(a) $\overline{C}_p(E;\Omega)=0$, and that

(b) $Pf = P(f+h)$ for all $f \in C(\partial\Omega)$ and $h:\partial\Omega \to \overline{\mathbf{R}}$ such that $h=0$ on $\partial\Omega\setminus E$?

A point $x \in \partial\Omega$ is *inaccessible* if there is no curve $\gamma:[0,1]\to\overline{\Omega}$ such that $\gamma([0,1))\subset\Omega$ and $\gamma(1)=x$. Here γ is *not* required to be rectifiable.

A positive answer to (a) directly yields a positive answer to (b), by Proposition 9.2.

In the linear case $p=2$ on unweighted $\mathbf{R}^n$, part (b) is true. This can be seen by observing that the Perron solution at a point $y\in\Omega$ is the expected value of the first point $x\in\partial\Omega$ which the Brownian motion (starting at y) hits, and this point is almost surely not in E. We are not aware of any nonprobabilistic proof of this fact.

In Example 10.2 we saw that both (a) and (b) are true for the topologist's comb in the nonlinear case. For the topologist's comb a more general invariance result is obtained in A. Björn [9].

11. Generalized Perron solutions for domains in Ω^M

We assume, in this section, that G is a bounded domain which is finitely connected at the boundary and that $\Omega\subset G$ is a nonempty open subset with $C_p(X\setminus\Omega)>0$. Recall also the standing assumptions from the end of Section 3.

Let us take another look at Examples 10.2 and 10.3. In Example 10.3 there are two directions of reaching every boundary point in the slits

$$A=\{0\}\times(0,1] \quad\text{and}\quad S_j^{\pm}=\{\pm 2^{-j}\}\times(0,1], \quad j=1,2,\ldots,$$

but since Ω is not finitely connected at the boundary we cannot use the Mazurkiewicz boundary results from Sections 7 and 8. A way around it is to consider Ω as a subdomain of a larger open set G, and equip Ω with the restriction of its Mazurkiewicz distance. In this section we consider such an approach. Since we now have to deal with two open sets, the notation becomes more cumbersome, which is why we avoided this generality in Sections 7 and 8. However, there are *no* additional technical difficulties.

We equip G with its Mazurkiewicz distance and consider its closure $\overline{G}^M$. This closure is compact by Theorem 4.5, since G is finitely connected at the boundary. Throughout this section the Mazurkiewicz distance is *always* taken with respect to G, and to avoid misunderstandings we write d_G instead of d_M. We also write Ω^G, $\overline{\Omega}^G$ and $\partial_{\overline{G}^M}\Omega$ when we equip Ω with the distance d_G and take its closure and boundary in $\overline{G}^M$.

Definition 11.1. Given a function $f : \partial_{\overline{G}^M}\Omega \to \overline{\mathbf{R}}$, let $\mathcal{U}_f(\Omega^G)$ be the set of all superharmonic functions u on Ω, bounded from below, such that

$$\liminf_{\Omega \ni y \xrightarrow{d_G} x} u(y) \geq f(x) \quad \text{for all } x \in \partial_{\overline{G}^M}\Omega.$$

The *generalized upper Perron solution* of f is defined by

$$\overline{P}_{\Omega^G} f(x) = \inf_{u \in \mathcal{U}_f(\Omega^G)} u(x), \quad x \in \Omega.$$

The *generalized lower Perron solution* $\underline{P}_{\Omega^G} f$ is defined similarly, or by $\underline{P}_{\Omega^G} f = -\overline{P}_{\Omega^G}(-f)$.

If $\overline{P}_{\Omega^G} f = \underline{P}_{\Omega^G} f$, then we let $P_{\Omega^G} f := \overline{P}_{\Omega^G} f$ and f is said to be *resolutive* with respect to Ω^G.

The results in Sections 7 and 8 can all be formulated in this generality, and the proofs remain the same. Let us formulate these results.

Theorem 11.2. *Assume that $h : \partial_{\overline{G}^M}\Omega \to \overline{\mathbf{R}}$ is zero $\overline{C}_p(\,\cdot\,;\Omega^G)$-q.e., If either $f \in C(\partial_{\overline{G}^M}\Omega)$, or $f : \overline{\Omega}^G \to \overline{\mathbf{R}}$ is a bounded $\overline{C}_p(\,\cdot\,;\Omega^G)$-quasicontinuous function such that $f|_\Omega \in N^{1,p}(\Omega)$, then f and $f+h$ are resolutive with respect to Ω^G and*

$$P_{\Omega^G}(f+h) = P_{\Omega^G} f = H_\Omega f.$$

Corollary 11.3. *Assume that either $f \in C(\partial_{\overline{G}^M}\Omega)$, or that $f : \overline{\Omega}^G \to \overline{\mathbf{R}}$ is a bounded $\overline{C}_p(\,\cdot\,;\Omega^G)$-quasicontinuous function such that $f|_\Omega \in N^{1,p}(\Omega)$. If u is a bounded p-harmonic function in Ω and if there is a set $E \subset \partial_{\overline{G}^M}\Omega$ with $\overline{C}_p(E;\Omega^G) = 0$ such that*

$$\lim_{\Omega \ni y \xrightarrow{d_G} x} u(y) = f(x) \quad \text{for all } x \in \partial_{\overline{G}^M}\Omega \setminus E,$$

then $u = P_{\Omega^G} f$.

As already mentioned, the proofs of these results are the same as the proofs given in Section 7. Let us just point out that the following fundamental equality follows from Proposition 5.3,

$$\begin{aligned} N_0^{1,p}(\Omega) &= \{f|_\Omega : f \in N_0^{1,p}(G) \text{ and } f = 0 \text{ in } G \setminus \Omega\} \\ &= \{f|_\Omega : f \in N_0^{1,p}(G^M) \text{ and } f = 0 \text{ in } G \setminus \Omega\} = N_0^{1,p}(\Omega; \overline{G}^M). \end{aligned}$$

Note also that the above results could not be obtained as direct consequences of the results in Section 8 by replacing X with $\overline{G}^M$ since the space $\overline{G}^M$ need not satisfy the standing assumptions about doubling and Poincaré inequality.

We end this section with a demonstration of the described technique in the case of the double comb. It makes it possible to treat discontinuities at finitely many slits S_j, including the central one A. This procedure can be iterated by adding more and more open slits, so that finally the whole comb can be treated, see A. Björn [9] for more details.

Example 11.4. Let $\Omega \subset \mathbf{R}^2$ be the double comb as in Example 10.3 and let

$$G = (-1,1)^2 \setminus (\overline{A} \cup \bigcup_{j\in J} \overline{S}_j^\pm),$$

where J is a finite set of indices and $A = \{0\}\times(0,1]$. Equip G with its Mazurkiewicz distance d_G and consider Ω as an open subset of G^M in this new metric. Observe that G is finitely connected at the boundary. The boundary $\partial_{\overline{G}^M}\Omega$ will be the same as $\partial_M\Omega$ apart from that the boundary points in $A\cup\bigcup_{j\in J} S_j^\pm$ will be split into two boundary points each.

Let f be the "jump" function from (10.2), and set $\tilde{f}=0$ on the left copy of A and $\tilde{f}=f$ otherwise. Then $\tilde{f}\in N^{1,p}(\overline{\Omega}^G)$ is continuous in $\overline{\Omega}^G$. Thus, it is resolutive with respect to Ω^G by Theorem 11.2, and can be perturbed arbitrarily on (both copies) of A without changing the obtained Perron solution. Since $\overline{C}_p(A,\Omega^G)=0$ and $f=\tilde{f}$ except at the left copy of A, the resolutivity of f with respect to Ω^G follows. Finally, it is easy to see that $P_\Omega f = P_{\Omega^G} f$, and thus f is resolutive also with respect to Ω.

Contrary to Example 10.3, this method also allows us to treat functions with different values on the left and right copies of the slits $S_j^\pm$, $j\in J$, e.g. similar to the "thickened" slits in Example 10.2.

Appendix. Comparison of capacities

In this appendix we do **not** *require the assumptions from the end of Section 3 to hold.*

The focus of this appendix is to compare the new capacity with the two natural capacities on $\overline{\Omega}$ equipped with μ resp. μ_0.

Proposition A.1. *If $E\subset\overline{\Omega}$, then*

$$C_p(E;(\overline{\Omega};\mu_0)) \le C_p(E;\overline{\Omega}) \le C_p(E).$$

This is trivial, and it is easy to find examples where strict inequalities hold by e.g. Examples 10.6 and 10.1 respectively.

For the new capacity the situation is a little more complicated.

Proposition A.2. *Let $G\subset\overline{\Omega}$ be relatively open. Then*

$$\overline{C}_p(G;\Omega) \le C_p(G;(\overline{\Omega};\mu_0)).$$

If X is proper and continuous functions are dense in $N^{1,p}(\overline{\Omega},\mu_0)$, then

$$\overline{C}_p(E;\Omega) \le C_p(E;(\overline{\Omega};\mu_0)) \quad \text{for all } E\subset\overline{\Omega}.$$

Proof. Since G is relatively open in $\overline{\Omega}$, every function admissible in the definition of $C_p(G;(\overline{\Omega},\mu_0))$ is also admissible for $\overline{C}_p(G;\Omega)$. Taking infimum over all such functions proves the first part.

As for the last part, we may assume that $C_p(E;(\overline{\Omega};\mu_0))<\infty$. Let $\varepsilon>0$. By Theorem 2.5 there is an open $G\supset E$ such that $C_p(G;(\overline{\Omega};\mu_0))<C_p(E;(\overline{\Omega};\mu_0))+\varepsilon$. By monotonicty and the first part we see that

$$\overline{C}_p(E;\Omega)\le\overline{C}_p(G;\Omega)\le C_p(G;(\overline{\Omega};\mu_0))<C_p(E;(\overline{\Omega};\mu_0))+\varepsilon.$$

Letting $\varepsilon\to 0$ concludes the proof. □

Example 10.2 (together with Proposition 3.3) shows that the inequality in Proposition A.2 can be strict. The following result shows that at the level of null sets Proposition A.2 holds for all sets without any continuity assumptions.

Proposition A.3. *Assume that X is proper. If $E\subset\overline{\Omega}$ and $C_p(E;(\overline{\Omega};\mu_0))=0$, then $\overline{C}_p(E;\Omega)=0$.*

Examples 10.2, 10.3 and 10.5 all show that the converse implication does not hold. Recall that if μ is doubling then X is complete if and only if X is proper.

Proof. Let $\varepsilon>0$. As $\overline{\Omega}$ is proper, Proposition 2.6 shows that there is a relatively open set $G\supset E$ with $C_p(G;(\overline{\Omega};\mu_0))<\varepsilon$. By Theorem A.2,

$$\overline{C}_p(E;\Omega)\le\overline{C}_p(G;\Omega)<\varepsilon.$$

Letting $\varepsilon\to 0$ completes the proof. □

The next proposition follows directly from Proposition A.2, since in the definition of quasicontinuity we only consider relatively open subsets of $\overline{\Omega}$.

Corollary A.4. *Assume that $f\in\overline{\Omega}\to\overline{\mathbf{R}}$ is quasicontinuous with respect to $(\overline{\Omega};\mu_0)$. Then f is $\overline{C}_p(\,\cdot\,;\Omega)$-quasicontinuous.*

We can also improve upon Lemma 5.2.

Proposition A.5. *Assume that X is proper and locally connected, and that Ω is a bounded domain which is finitely connected at the boundary. Let $E\subset\overline{\Omega}$. Then*

$$\overline{C}_p(\Phi^{-1}(E);\Omega^M)=\overline{C}_p(E;\Omega).$$

Proof. The inequality $\overline{C}_p(\Phi^{-1}(E);\Omega^M)\le\overline{C}_p(E;\Omega)$ was proved in Lemma 5.2. (Note that the proof of the first inequality in Lemma 5.2 only requires X to be locally connected, through the equality $N^{1,p}(\Omega)=N^{1,p}(\Omega^M)$.) For the converse inequality, assume that $\overline{C}_p(\Phi^{-1}(E);\Omega^M)<\infty$ and let $u\in\mathcal{A}_{\Phi^{-1}(E)}$. For $x\in E\cap\partial\Omega$ let $x_j\in\Omega$, $j=1,2,...$, be a sequence of points such that $x_j\to x$ in the metric d as $j\to\infty$.

We shall show that $\liminf_{j\to\infty}u(x_j)\ge 1$. Assume not. Then there is a subsequence, also denoted $\{x_j\}_{j=1}^\infty$, such that $\lim_{j\to\infty}u(x_j)<1$. By the compactness of $\overline{\Omega}^M$ (see Theorem 4.5 and the comment after it), the sequence $\{x_j\}_{j=1}^\infty$ has a convergent subsequence $\{x_{j_k}\}_{k=1}^\infty$ tending in the metric d_M to some point $x_0\in\overline{\Omega}^M$. As Φ is Lipschitz on $\overline{\Omega}^M$, we have

$$\Phi(x_0)=\lim_{k\to\infty}\Phi(x_{j_k})=\lim_{k\to\infty}x_{j_k}=x,$$

i.e. $x_0\in\Phi^{-1}(E)$. Since $u\in\mathcal{A}_{\Phi^{-1}(E)}$, it follows that

$$\liminf_{k\to\infty}u(x_{j_k})\ge 1,$$

contradicting $\lim_{j\to\infty} u(x_j) < 1$. Thus $\liminf_{j\to\infty} u(x_j) \geq 1$ and $u \in \mathcal{A}_E$ (observe that $N^{1,p}(\Omega) = N^{1,p}(\Omega^M)$ by the discussion at the beginning of Section 5.). Hence

$$\overline{C}_p(E;\Omega) \leq \|u\|^p_{N^{1,p}(\Omega)}$$

and taking infimum over all $u \in \mathcal{A}_{\Phi^{-1}(E)}$ completes the proof. □

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